arXiv:funct-an/9604011v1 30 Apr 1996

# On the multiplication of free $n$-tuples of non-commutative random variables

Alexandru Nica *
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1003, USA
(e-mail: andu@math.lsa.umich.edu)

Roland Speicher †
Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 294
D-69120 Heidelberg, Germany
(e-mail: roland.speicher@urz.uni-heidelberg.de)

**Abstract**

Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be random variables in some (non-commutative) probability space, such that $\{a_1, \ldots, a_n\}$ is free from $\{b_1, \ldots, b_n\}$. We show how the joint distribution of the $n$-tuple $(a_1 b_1, \ldots, a_n b_n)$ can be described in terms of the joint distributions of $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, by using the combinatorics of the $n$-dimensional $R$-transform. We point out a few applications that can be easily derived from our result, concerning the left-and-right translation with a semicircular element (see Sections 1.6-1.10) and the compression with a projection (see Sections 1.11-1.14) of an $n$-tuple of non-commutative random variables. A different approach to two of these applications is presented by Dan Voiculescu in an Appendix to the paper.

*Research done while this author was on leave at the Fields Institute, Waterloo, and the Queen's University, Kingston, holding a Fellowship of NSERC, Canada.

†Supported by a Heisenberg Fellowship of the DFG.

## Introduction

The theory of free random variables was developed in a sequence of papers of D. Voiculescu (see [17], or the recent survey in [16]), as an instrument for approaching free products of operator algebras. Its particular aspect addressed in the present paper is the one concerning the addition and multiplication of free random variables; as shown by Voiculescu in [13], [14], a powerful method in the study of these operations is by the use of *transforms* that convert them (respectively) into addition and multiplication of complex analytic functions - or, in an algebraic framework, of formal power series in an indeterminate $z$. The precise definitions of these transforms (called *R-transform* for the addition problem and *S-transform* for the multiplication problem) will be reviewed in the Sections 1.2, 1.3 below.

In the present paper we are pursuing a combination of two ideas that have appeared recently in the study of the $R$- and $S$-transforms.

The first idea is that the connection between the $R$- and the $S$-transform is closer than one might suspect at first glance. A way of making this precise was pointed out in our paper [6], in the form of the equation $S(\mu) = \mathcal{F}(R(\mu))$, for $\mu$ a distribution with non-vanishing mean, and where $\mathcal{F}$ is a combinatorial object with a precise significance ("the Fourier transform for multiplicative functions on non-crossing partitions"). A byproduct of our result in [6] is the remark that the multiplication of free random variables can be studied *directly in terms of the R-transform*, via an equation of the form $R(\mu_{ab}) = R(\mu_a) \boxed{\star} R(\mu_b)$, where $a, b$ are free random variables in some non-commutative probability space, $\mu_a, \mu_b, \mu_{ab}$ stand for the distributions of $a, b$ and $ab$, respectively, and where the operation $\boxed{\star}$ is again an object with precise combinatorial significance, "the convolution of multiplicative functions on non-crossing partitions".

The second idea, appearing in [8], [5], is that the $R$-transform has natural *multidimensional analogues*. Besides handling the addition of free $n$-tuples ($n \geq 1$) of random variables, the multidimensional $R$-transform has the important property that

$$(I) \qquad [R(\mu_{a'_1,\ldots,a'_m,a''_1,\ldots,a''_n})](z'_1,\ldots,z'_m,z''_1,\ldots,z''_n) =$$

$$= [R(\mu_{a'_1,\ldots,a'_m})](z'_1,\ldots,z'_m) + [R(\mu_{a''_1,\ldots,a''_n})](z''_1,\ldots,z''_n),$$

for every family $a'_1,\ldots,a'_m,a''_1,\ldots,a''_n$ of random variables in some non-commutative probability space, such that $\{a'_1,\ldots a'_m\}$ is free from $\{a''_1,\ldots,a''_n\}$. (In Equation $(I)$, by e.g. $\mu_{a'_1,\ldots,a'_m}$ we understand the joint distribution of the variables $a'_1,\ldots,a'_m$, and $R(\mu_{a'_1,\ldots,a'_m})$ is a formal power series in the $m$ non-commuting variables $z'_1,\ldots,z'_m$.) This property opens a

whole new array of possibilities, in the direction of analyzing freeness for families of random variables by the use of an *R-transform calculus,* i.e. of a formal manipulation of power series which lands with an equation of the type of $(I)$.

Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be random variables in some (non-commutative) probability space, such that $\{a_1, \ldots, a_n\}$ is free from $\{b_1, \ldots, b_n\}$. In the main theorem of the present paper we show how the joint distribution of the $n$-tuple $(a_1b_1, \ldots, a_nb_n)$ can be described in terms of the joint distributions of $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, *by using the $n$-dimensional $R$-transform.* The formula we obtain is

$$(II) \qquad R(\mu_{a_1b_1,\ldots,a_nb_n}) \ = \ R(\mu_{a_1,\ldots,a_n}) \boxed{\star} R(\mu_{b_1,\ldots,b_n}),$$

where, of course, the crucial point is that the operation $\boxed{\star}$ can be well understood combinatorially (it extends the "convolution of multiplicative functions on non-crossing partitions", mentioned above in the 1-dimensional case).

In the support of the idea that the formula $(II)$ can be really useful in approaching the multiplication of free $n$-tuples, we present four applications that can be easily proved from it, on the lines of the "$R$-transform calculus" mentioned in the paragraph containing Equation $(I)$.

Two of these applications are concerning the left-and-right translation of a family of random variables by a centered semicircular element which is free from the family. The phenomenon here is that, roughly, this operation "converts orthogonality into freeness" (see Corollary 1.8 below). For instance, in the important case when $e_1, \ldots, e_n$ are pairwisely orthogonal projections in a tracial non-commutative probability space $(\mathcal{A}, \varphi)$, and $b \in \mathcal{A}$ is a centered semicircular element of radius $r$ which is free from $\{e_1, \ldots, e_n\}$, one gets that $\{be_1b, \ldots, be_nb\}$ is a free family, and that $be_kb$ is a Poisson element of parameters $\varphi(e_k)$ and $r^2/4$, $1 \leq k \leq n$ (see Remark 1.9). Instead of "left-and-right translation with a semicircular element" one can use in this result "conjugation with a circular element" or "left-and-right translation with a quarter-circular element" (Remark 1.7). Moreover, we point out the fact that if $b$ is a centered semicircular element free from the family $\{a_1, \ldots, a_n\}$, then, without any additional assumption on $a_1, \ldots, a_n$, one always gets that $\{ba_1b, \ldots, ba_nb\}$ is free from $\{a_1, \ldots, a_n\}$ (Application 1.10).

The other two applications are related to the compression of a family of random variables by a projection which is free from the family. The phenomenon here is, roughly, that "freeness is preserved by the compression" (see Corollary 1.12 and Application 1.13 below).

In the 1-dimensional case it is interesting to note that, given a probability measure $\mu$ with compact support on $\mathbf{R}$, one gets a realization of the semigroup of measures $(\mu_t)_t$ having $R(\mu_t) = tR(\mu)$ which was studied by Bercovici and Voiculescu in [1]; namely, $\mu_t$ is obtained by essentially "compressing $\mu$ with a projection free from $\mu$ and of trace $1/t$." This shows in particular that the semigroup $(\mu_t)_t$ can always be started at $t = 1$.

The paper is divided into sections as follows. In Section 1 we make a detailed presentation of the results announced above, after reviewing the basic free probabilistic concepts that we are using. Section 2 is devoted to some combinatorial preliminaries about non-crossing partitions. The operation $\boxed{\star}$ (mentioned in Eqn.$(II)$ above) is introduced and discussed in Section 3, where the main result of the paper, Theorem 1.4, is also proved. The four applications mentioned above (which are stated precisely in Sections 1.6, 1.10, 1.11, 1.13) have their proofs presented in the final Section 4.

The work presented here was started during a workshop on operator algebra free products and random matrices held at the Fields Institute, Waterloo, in March 1995. We would like to thank the organizers for the very stimulating and inspiring atmosphere that animated the workshop.

Also, we would like to acknowledge several useful discussions with Dan Voiculescu concerning our work. It turned out that the realization of the free Poisson variables mentioned above had been known to him for some time (although not published); and that he could give a different proof for the preservation of freeness among non-commutative random variables, under compression with a free projection. His approach is presented in an Appendix to the present paper.

## 1. Presentation of the results

In order to make the presentation self-contained, we begin by reviewing a few basic definitions concerning free random variables; for more details, the reader is referred to the monograph [17].

**1.1 Basic definitions** We will consider a purely algebraic framework, where considerations on the positivity or measurability of the random variables involved aren't neces-

sarily required; thus by a *non-commutative probability space* we will simply understand a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a complex unital algebra ("the algebra of random variables") and $\varphi : \mathcal{A} \to \mathbf{C}$ ("the expectation") is a linear functional, normalized by $\varphi(1) = 1$. The unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{A}$ are called *free* (with respect to the expectation $\varphi$) if for every $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$ and $a_1 \in \mathcal{A}_{i_1}, \ldots, a_k \in \mathcal{A}_{i_k}$ such that:
(i) $i_{j-1} \neq i_j$ for $1 \leq j \leq k-1$, and
(ii) $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_k) = 0$
it follows that $\varphi(a_1 a_2 \cdots a_k) = 0$. The definition of freeness extends to arbitrary subsets of $\mathcal{A}$, by defining $\mathcal{X}_1, \ldots, \mathcal{X}_k \subseteq \mathcal{A}$ to be free whenever the unital algebras generated by them are so.

If $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and if $a_1, \ldots, a_n$ are elements ("random variables") in $\mathcal{A}$, then the *joint distribution* of $a_1, \ldots, a_n$ is by definition the linear functional $\mu_{a_1,\ldots,a_n} : \mathbf{C}\langle X_1, \ldots, X_n\rangle \to \mathbf{C}$ determined by

$$\begin{cases} \mu_{a_1,\ldots,a_n}(1) = 1 \\ \mu_{a_1,\ldots,a_n}(X_{i_1} X_{i_2} \cdots X_{i_k}) \ = \ \varphi(a_{i_1} a_{i_2} \cdots a_{i_k}), \text{ for } k \geq 1 \text{ and } 1 \leq i_1, \ldots, i_k \leq n, \end{cases} \tag{1.1}$$

where $\mathbf{C}\langle X_1, \ldots, X_n\rangle$ denotes the algebra of polynomials in $n$ non-commuting indeterminates $X_1, \ldots, X_n$. In the case $n = 1$, the Equation (1.1) defines the *distribution* of the single element $a = a_1 \in \mathcal{A}$ (which is the linear functional $\mu_a : \mathbf{C}[X] \to \mathbf{C}$ determined by $\mu_a(X^n) = \varphi(a^n)$, $n \geq 0$).

An important example of distribution is the *semicircle law*, which plays in free probability a role analogous to the one of Gaussian measures in classical probability. If $(\mathcal{A}, \varphi)$ is a non-commutative probability space, an element $a \in \mathcal{A}$ will be called *centered semicircular* of radius $r > 0$ if its distribution is determined by

$$\mu_a(X^n) \ = \ \varphi(a^n) \ = \ \frac{2}{\pi r^2} \int_{-r}^{r} t^n \sqrt{r^2 - t^2} \, dt, \quad n \geq 0. \tag{1.2}$$

In Sections 1.7, 1.8 below we will also meet the situation of a non-commutative probability space $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital $\star$-algebra and $\varphi$ has the property that $\varphi(a^*) = \overline{\varphi(a)}$, for every $a \in \mathcal{A}$; this is called a (non-commutative) *$\star$-probability space.* When giving the definition of a centered semicircular element $a$ in a $\star$-probability space, one also adds to (1.2) the condition that $a = a^*$.

**1.2 The $R$-transform** ( [13], [8], [5]) is a useful tool for studying joint distributions of free families of random variables. In order not to divagate too much from the main stream of our presentation, we defer for the moment the review of the precise definition of

the $R$-transform (see Section 3.9 below), and only mention here the nature of this object; namely, for every $n \geq 1$, the $n$-dimensional $R$-transform is a certain bijection $R$ from the set of linear functionals $\Sigma_n = \{\mu : \mathbf{C}\langle X_1, \ldots, X_n \rangle \to \mathbf{C} \mid \mu \text{ linear}, \mu(1) = 1\}$ onto the set $\Theta_n$ of formal power series in $n$ non-commuting variables, and without constant coefficient. (An arbitrary element of $\Theta_n$ is thus a series of the form

$$f(z_1, \ldots, z_n) \;=\; \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{n} \alpha_{(i_1, \ldots, i_k)} z_{i_1} \cdots z_{i_k}, \tag{1.3}$$

with $(\alpha_{(i_1,\ldots,i_k)})_{k \geq 1, 1 \leq i_1, \ldots, i_k \leq n}$ complex coefficients.)

The main property of the $R$-transform is the following: if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and $a_1', \ldots a_m', a_1'', \ldots, a_n'' \in \mathcal{A}$ are such that $\{a_1', \ldots a_m'\}$ [1] is free from $\{a_1'', \ldots, a_n''\}$, then

$$[R(\mu_{a_1',\ldots,a_m',a_1'',\ldots,a_n''})](z_1', \ldots, z_m', z_1'', \ldots, z_n'') \;= \tag{1.4}$$

$$=\; [R(\mu_{a_1',\ldots,a_m'})](z_1', \ldots, z_m') + [R(\mu_{a_1'',\ldots,a_n''})](z_1'', \ldots, z_n'');$$

and conversely, the fact that $R(\mu_{a_1',\ldots,a_m',a_1'',\ldots,a_n''})$ "has no cross-terms" (i.e. it is the sum between a series in $z_1', \ldots, z_m'$ and a series in $z_1'', \ldots, z_n''$) is sufficient to ensure that $\{a_1', \ldots a_m'\}$ is free from $\{a_1'', \ldots, a_n''\}$. Clearly, this property can make the $R$-transform very useful for analyzing freeness, in the situations when we have the capability of calculating it explicitly.

The $R$-transform was first remarked by Voiculescu ([12], [13]) in the one-dimensional case, for another important property:

$$[R(\mu_{a+b})](z) \;=\; [R(\mu_a)](z) + [R(\mu_b)](z), \tag{1.5}$$

whenever $a, b$ are free in some non-commutative probability space $(\mathcal{A}, \varphi)$. This property can be shown ([8], [5]) to hold in any dimension, i.e.:

$$[R(\mu_{a_1+b_1,\ldots,a_n+b_n})](z_1, \ldots, z_n) \;=\; [R(\mu_{a_1,\ldots,a_n})](z_1, \ldots, z_n) + [R(\mu_{b_1,\ldots,b_n})](z_1, \ldots, z_n), \tag{1.6}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$, and $\{a_1, \ldots, a_n\}$ is free from $\{b_1, \ldots, b_n\}$. Equation (1.6) shows that, in any case, the $R$-transform is well-suited for studying componentwise *sums* of free $n$-tuples of random variables.

[1] We allow the possibility that there exist $i \neq j$ such that $a_i' = a_j'$; $\{a_1', \ldots a_m'\}$ denotes here the set obtained from $a_1', \ldots, a_m'$ after deleting the repetitions.

**1.3 The $S$-transform** As mentioned in the introduction, our goal in the present note is to study componentwise *products* of free $n$-tuples. The case $n = 1$ comes to considering the product of two free random variables, and was analyzed by Voiculescu in [14] by means of the $S$-transform. This is a bijective map $S$ from the set of linear functionals $\{\mu : \mathbf{C}[X] \to \mathbf{C} \mid \mu$ linear, $\mu(1) = 1, \mu(X) \neq 0\}$ onto the set of formal power series $\{g \mid g(z) = \sum_{n=0}^{\infty} \beta_n z^n$, $\beta_0, \beta_1, \beta_2, \ldots \in \mathbf{C}, \beta_0 \neq 0\}$; it can be described by the formula ([14], Theorem 2.6):

$$[S(\mu)](z) \;=\; \frac{1+z}{z} (\sum_{n=1}^{\infty} \mu(X^n) z^n)^{<-1>}, \tag{1.7}$$

where $^{<-1>}$ denotes the inverse under the operation of composition of formal power series. The $S$-transform has the "multiplicative analogue" of the property mentioned in Eqn.(1.5), i.e.

$$[S(\mu_{ab})](z) \;=\; [S(\mu_a)](z) \cdot [S(\mu_b)](z), \tag{1.8}$$

whenever $a, b$ are free random variables in some non-commutative probability space $(\mathcal{A}, \varphi)$, such that $\varphi(a) \neq 0 \neq \varphi(b)$.

In [6] we have found an alternative proof of (1.8), which has the interesting feature that it goes by relating the $S$-transform to the (1-dimensional) $R$-transform. The main steps of the argument can be presented as follows:

(a) Consider the range-set $\Theta_1 = \{f \mid f(z) = \sum_{n=1}^{\infty} \alpha_n z^n,\ \alpha_1, \alpha_2, \ldots \in \mathbf{C}\}$ of the 1-dimensional $R$-transform; we put into evidence a binary operation on $\Theta_1$, which will be denoted here by $\boxed{\star}$, and has the property that

$$R(\mu_{ab}) \;=\; R(\mu_a) \boxed{\star} R(\mu_b) \tag{1.9}$$

whenever $a, b$ are free random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$ (the condition $\varphi(a) \neq 0 \neq \varphi(b)$ is not required at this stage).

(b) We put into evidence a bijection $\mathcal{F}$ from $\{f \mid f(z) = \sum_{n=1}^{\infty} \alpha_n z^n,\ \alpha_1, \alpha_2, \ldots \in \mathbf{C}, \alpha_1 \neq 0\}$ onto $\{g \mid g(z) = \sum_{n=0}^{\infty} \beta_n z^n,\ \beta_0, \beta_1, \beta_2, \ldots \in \mathbf{C}, \beta_0 \neq 0\}$, such that

$$[\mathcal{F}(f' \boxed{\star} f'')](z) \;=\; [\mathcal{F}(f')](z) \cdot [\mathcal{F}(f'')](z) \tag{1.10}$$

for every $f', f''$ in the domain of $\mathcal{F}$.

(c) We show that

$$S(\mu_a) \;=\; \mathcal{F}(R(\mu_a)) \tag{1.11}$$

for every random variable $a$ in a non-commutative probability space $(\mathcal{A}, \varphi)$, with $\varphi(a) \neq 0$.

(a),(b) and (c) above are clearly implying (1.8), since (for $a, b$ as in (1.8)) we have

$$S(\mu_{ab}) \overset{(1.11)}{=} \mathcal{F}(R(\mu_{ab})) \overset{(1.9)}{=} \mathcal{F}(R(\mu_a) \boxed{\star} R(\mu_b)) \overset{(1.10)}{=} \mathcal{F}(R(\mu_a)) \cdot \mathcal{F}(R(\mu_b)) \overset{(1.11)}{=} S(\mu_a) \cdot S(\mu_b).$$

The framework in [6] is combinatorial, the object of study being "the lattice of non-crossing partition of a finite ordered set" (notion recalled in Section 2 below). It is important to mention that the operation $\boxed{\star}$ appearing in Eqn.(1.9) has a clear significance in this combinatorial context, it is "the convolution of multiplicative functions on non-crossing partitions" ([6], Section 1.4).

Now, while approaching products of free $n$-tuples of random variables via an $n$-dimensional version of the $S$-transform seems to be difficult (or even only partly possible - see also Remark 3.12 below), the point we would like to make is that the operation $\boxed{\star}$ of Eqn.(1.9) *does extend naturally* to the $n$-dimensional situation, and allows us to gain information on products of free $n$-tuples *via the $n$-dimensional $R$-transform.* More precisely, we have:

**1.4 Theorem** Let $n$ be a positive integer. There exists a binary operation, $\boxed{\star}$, on the space of formal power series of the form shown in (1.3), which is defined via certain "summation formulae on non-crossing partitions", [2] and has the following property: if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ are such that $\{a_1, \ldots, a_n\}$ is free from $\{b_1, \ldots, b_n\}$, then

$$R(\mu_{a_1 b_1, \ldots, a_n b_n}) = R(\mu_{a_1, \ldots, a_n}) \boxed{\star} R(\mu_{b_1, \ldots, b_n}). \tag{1.12}$$

The crucial point in the above Theorem is that the operation $\boxed{\star}$ *has a precise combinatorial significance* (merely establishing the existence of an operation with the property (1.12) would be trivial, but also of no use). In the support of the idea that the result in 1.4 can be really useful in approaching products of free $n$-tuples, we present a few applications that can be easily derived from it.

The first of the applications is concerning the left-and-right translation of a family of random variables by a centered semicircular element which is free from the family. We will use the following

[2] The exact definition of $\boxed{\star}$ will be given in Section 3.2, after the necessary definitions concerning non-crossing partitions are reviewed in Section 2.

**1.5 Notation:** For $n \geq 1$ and $\mu : \mathbf{C}\langle X_1, \ldots, X_n\rangle \to \mathbf{C}$ a linear functional normalized by $\mu(1) = 1$, we denote by $M(\mu)$ the formal power series in $n$ non-commuting variables which has the moments of $\mu$ as coefficients, i.e.

$$[M(\mu)](z_1, \ldots, z_n) \ = \ \sum_{k=1}^{\infty} \ \sum_{i_1,\ldots,i_k=1}^{n} \mu(X_{i_1} \cdots X_{i_k}) z_{i_1} \cdots z_{i_k}. \tag{1.13}$$

**1.6 Application** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space such that $\varphi$ is a trace (i.e. $\varphi(xy) = \varphi(yx)$ for all $x, y \in \mathcal{A}$). Let $a_1, \ldots, a_n, b \in \mathcal{A}$ be such that:
(i) $b$ is a centered semicircular element of radius $r$ (in the sense reviewed in 1.1); and
(ii) $\{a_1, \ldots, a_n\}$ is free from $b$. Then

$$R(\mu_{ba_1b,\ldots,ba_nb}) \ = \ M(\mu_{\frac{r^2}{4}a_1,\ldots,\frac{r^2}{4}a_n}). \tag{1.14}$$

**1.7 Remark** The first step in deriving (1.14) is to note that $\mu_{ba_1b,\ldots,ba_nb} = \mu_{a_1b^2,\ldots,a_nb^2}$. If $b$ is centered semicircular, then $b^2$ is what is called "a Poisson element" (see e.g. [17], the comment in [3] (c) following to Theorem 3.7.2); hence Application 1.6 can be viewed as concerning right (or equivalently, left) translations with this Poisson element.

Moreover, it is known that the same Poisson element can be obtained:

- either as $q^2$, with $q$ "quarter-circle element of radius $r$" (i.e. having $\varphi(q^n) = \frac{4}{\pi r^2} \int_0^r t^n \sqrt{r^2 - t^2} \, dt$ for $n \geq 0$ - see [17], Definition 5.1.9);

- or as $c^*c$, with $c$ "circular element of radius $r$", i.e. of the form $c = (x + iy)/\sqrt{2}$ with $x, y$ free centered semicircular elements of radius $r$ (see [15], Definition 1.9 or [17], Definition 5.1.1; this makes of course sense only if $(\mathcal{A}, \varphi)$ is a $\star$-probability space).

Consequently, one can also replace the left-hand side of (1.14) by $R(\mu_{qa_1q,\ldots,qa_nq})$ or by $R(\mu_{ca_1c^*,\ldots,ca_nc^*})$, with $q$ and $c$ as above.

**1.8 Corollary** Let $(\mathcal{A}, \varphi)$ be a non-commutative $\star$-probability space, and let $a_1, \ldots, a_n, b$ be selfadjoint elements of $\mathcal{A}$ such that:

(i) $b$ is a centered semicircular element;

[3] A random variable in a non-commutative probability space is called Poisson of parameters $\alpha, \beta$ if the $R$-transform of its distribution is $\alpha\beta z/(1 - \beta z)$; the situation encountered here is the one corresponding to the case $\alpha = 1, \ \beta = r^2/4$, with $r$ the radius of $b$. We mention that in the present note we have taken the liberty of multiplying by $z$ the $R$-series as used in [17], because this makes the notations easier when passing to the multidimensional case.

(ii) $a_i a_j = 0$ for $i \neq j$ (e.g., $a_1, \ldots, a_n$ can be mutually orthogonal projections);

(iii) $\{a_1, \ldots, a_n\}$ is free from $b$.

Then $ba_1b, \ldots, ba_nb$ form a free family in $(\mathcal{A}, \varphi)$.

[Alternative statements, following from Remark 1.7: one can replace "$b$ semicircular" by "$q$ quarter-circular" or by "$c$ circular", where in the latter version $c$ isn't required to be selfadjoint, and the $n$-tuple in the conclusion is $ca_1c^*, \ldots, ca_nc^*$.]

**Proof** We may assume that $\varphi$ is a trace (because in (iii) we are having two free Abelian subalgebras of $\mathcal{A}$, and by Proposition 2.5.3 of [17]). Then

$$[R(\mu_{ba_1b,\ldots,ba_nb})](z_1, \ldots, z_n) = [M(\mu_{\frac{r^2}{4}a_1,\ldots,\frac{r^2}{4}a_n})](z_1, \ldots, z_n) \quad \text{(by (1.14))}$$

$$= \sum_{m=1}^{n} (\sum_{k=1}^{\infty} (r^2/4)^k \varphi(a_m^k) z_m^k) \quad \text{(by hypothesis (ii));}$$

hence $R(\mu_{ba_1b,\ldots,ba_nb})$ has no cross-terms (in the sense reviewed in 1.2), and $ba_1b, \ldots, ba_nb$ must be free. **QED**

**1.9 Remark** In addition to the statement of 1.8, note that the individual distributions of the elements $ba_1b, \ldots, ba_nb$ can be effectively calculated by using the 1-dimensional case of Eqn.(1.14); more precisely, we have $[R(\mu_{ba_kb})](z) = [M(\mu_{a_k})](\frac{r^2}{4}z)$, $1 \leq k \leq n$, where $r$ is the radius of $b$. In particular, if the moments of $a_k$ are given by "a nice formula", the same will happen with the $R$-transform of $ba_kb$. For instance, if $a_k$ is a projection of trace $\alpha$, then $[M(\mu_{a_k})](z) = \alpha z/(1-z)$, hence $[R(\mu_{ba_kb})](z) = (\alpha\frac{r^2}{4}z)/(1-\frac{r^2}{4}z)$, i.e. $ba_kb$ is a Poisson element of parameters $\alpha$ and $r^2/4$.

In the context of Sections 1.6-1.9, let us finally point out that we also have:

**1.10 Application** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space such that $\varphi$ is a trace, and let $a_1, \ldots, a_n, b \in \mathcal{A}$ be such that $b$ is a centered semicircular element, free from $\{a_1, \ldots, a_n\}$. Then $\{ba_1b, \ldots, ba_nb\}$ is free from $\{a_1, \ldots, a_n\}$.

The other two applications we present are related to the compression by a projection.

**1.11 Application** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, such that $\varphi$ is a trace. Consider an idempotent $p \in \mathcal{A}$, and denote $\varphi(p) \stackrel{def}{=} \alpha$; we assume that $\alpha \neq 0$. If $a_1, \ldots, a_n \in \mathcal{A}$ are such that $\{a_1, \ldots, a_n\}$ is free from $p$ in $(\mathcal{A}, \varphi)$, then

$$R(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np}) \;=\; \frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{\alpha a_1,\ldots,\alpha a_n}), \tag{1.15}$$

where $\mu^{(p\mathcal{A}p)}_{\cdots}$ means that the corresponding joint distribution is considered in the non-commutative probability space $(p\mathcal{A}p, \frac{1}{\alpha}\varphi|p\mathcal{A}p)$ (while $\mu^{(\mathcal{A})}_{\cdots}$ means that we have a joint distribution considered in $(\mathcal{A}, \varphi)$).

**1.12 Corollary** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, such that $\varphi$ is a trace, and let $p \in \mathcal{A}$ be an idempotent with $\varphi(p) = \alpha \neq 0$. If $a_1, \ldots, a_n \in \mathcal{A}$ is a free family of elements of $\mathcal{A}$, such that $\{a_1, \ldots, a_n\}$ is free from $p$, then $pa_1p, \ldots, pa_np$ is a free family in $(p\mathcal{A}p, \frac{1}{\alpha}\varphi|p\mathcal{A}p)$.

**Proof** From (1.15) and the freeness of $\alpha a_1, \ldots, \alpha a_n$ it is immediate that $R(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np})$ has no cross-terms; hence $pa_1p, \ldots, pa_np$ must be free in $p\mathcal{A}p$. **QED**

The instance of the Corollary 1.12 when $a_1, \ldots, a_n$ are centered semicircular random variables was derived in [15, Proposition 2.3] by using arguments of random matrices; in this case, the compressions $pa_1p, \ldots, pa_np$ are also centered semicircular. (Note that if an $a_k$, $1 \leq k \leq n$, appearing in 1.12 is centered semicircular of radius $r$ in $(\mathcal{A}, \varphi)$, then $pa_kp$ is centered semicircular of radius $r\sqrt{\alpha}$ in $(p\mathcal{A}p, \frac{1}{\alpha}\varphi|p\mathcal{A}p)$; this is by Eqn.(1.15) applied in the 1-dimensional case, and the known fact that the $R$-transform of a centered semicircular element of radius $r$ is $\frac{r^2}{4}z^2$ - see [17], Example 3.4.4).

In [15], Proposition 2.3 it is also remarked that if the idempotent $p$ is picked in an Abelian unital subalgebra $\mathcal{D} \subseteq \mathcal{A}$, which is free from $\{a_1, \ldots, a_n\}$, then $p\mathcal{D}p$ is free from $\{pa_1p, \ldots, pa_np\}$ in $p\mathcal{A}p$. This latter assertion doesn't follow from Application 1.11, but it turns out that it can still be derived via calculations involving the operation $\boxed{\star}$ of Theorem 1.4; moreover, the assumption that $\mathcal{D}$ is commutative can be dropped - that is:

**1.13 Application** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space such that $\varphi$ is a trace. Let $\mathcal{B} \subseteq \mathcal{A}$ be a unital subalgebra, and let $p \in \mathcal{B}$ be an idempotent such that $\varphi(p) = \alpha \neq 0$. If $a_1, \ldots, a_n \in \mathcal{A}$ are such that $\{a_1, \ldots, a_n\}$ is free from $\mathcal{B}$ in $(\mathcal{A}, \varphi)$, then $\{pa_1p, \ldots, pa_np\}$ is free from $p\mathcal{B}p$ in $(p\mathcal{A}p, \frac{1}{\alpha}\varphi|p\mathcal{A}p)$.

We conclude this section by noting that the 1-dimensional version of Application 1.11 also has the following interesting consequence.

**1.14 Corollary** Let $\mu$ be a compactly supported probability measure on $\mathbf{R}$; then for every $t \geq 1$ there exists a unique compactly supported probability measure $\mu_t$ on $\mathbf{R}$, such that $R(\mu_t) = tR(\mu)$.
[See also [1], Proposition 8, where the semigroup $(\mu_t)_t$ is shown to exist and exhibit strong properties, for $t$ sufficiently large.]

**Proof** It is easy to find a von Neumann algebra $\mathcal{A}$ with a normal trace-state $\varphi : \mathcal{A} \to \mathbf{C}$, and $a = a^* \in \mathcal{A}$, $(p_\alpha)_{0 \leq \alpha \leq 1}$ selfadjoint projections in $\mathcal{A}$, such that: (i) the distribution of $a$ in $(\mathcal{A}, \varphi)$ is $\mu$; (ii) $\varphi(p_\alpha) = \alpha$, for every $0 \leq \alpha \leq 1$; (iii) $a$ is free from $\{p_\alpha \mid 0 \leq \alpha \leq 1\}$ (these elements can be realized for instance in the free product between $L^\infty(\mu)$ and the $L^\infty$-algebra of the Lebesgue measure on [0,1]). For every $t \geq 1$ we consider the selfadjoint element $a_t = tp_{1/t}ap_{1/t} \in p_{1/t}\mathcal{A}p_{1/t}$. The distribution $\mu_t$ of $a_t$ in $p_{1/t}\mathcal{A}p_{1/t}$ is a compactly supported measure on $\mathbf{R}$ (see [17], Remark 2.3.2), and has $R(\mu_t) = tR(\mu)$ by Eqn.(1.15). The uniqueness of $\mu_t$ is clear, since $R(\mu_t) = tR(\mu)$ determines the moments of $\mu_t$. **QED**

## 2. Preliminaries about non-crossing partitions

**2.1 Definitions** $1^o$ If $\pi = \{B_1, \ldots, B_k\}$ is a partition of $\{1, \ldots, n\}$ (i.e. $B_1, \ldots, B_k$ are pairwisely disjoint, non-void sets, such that $B_1 \cup \cdots \cup B_k = \{1, \ldots, n\}$), then the equivalence relation on $\{1, \ldots, n\}$ with equivalence classes $B_1, \ldots, B_k$ will be denoted by $\overset{\pi}{\sim}$; the sets $B_1, \ldots, B_k$ will be also referred to as the *blocks* of $\pi$. The number of elements in the block $B_j$, $1 \leq j \leq k$, will be denoted by $|B_j|$.

A partition $\pi$ of $\{1, \ldots, n\}$ is called *non-crossing* if for every $1 \leq i < j < k < l \leq n$ such that $i \overset{\pi}{\sim} k$ and $j \overset{\pi}{\sim} l$, it necessarily follows that $i \overset{\pi}{\sim} j \overset{\pi}{\sim} k \overset{\pi}{\sim} l$. The set of non-crossing partitions of $\{1, \ldots, n\}$ will be denoted by $NC(n)$.

For instance, all the partitions of $\{1, \ldots, n\}$ with $n \leq 3$ are non-crossing, and the only partition of $\{1, 2, 3, 4\}$ which is not non-crossing is $\{\{1, 3\}, \{2, 4\}\}$. In general, the number of non-crossing partitions of $\{1, \ldots, n\}$ can be shown (see e.g. [4]) to be the Catalan number $(2n)!/(n!(n+1)!)$.

$2^0$ For $\pi, \rho \in NC(n)$, we will write "$\pi \leq \rho$" if each block of $\rho$ is a union of blocks of $\pi$ (equivalently, if we have the implication "$i \overset{\pi}{\sim} j \Rightarrow i \overset{\rho}{\sim} j$", $1 \leq i, j \leq n$). Then "$\leq$" is a partial order relation on $NC(n)$ (called the *refinement order*), and $(NC(n), \leq)$ can in fact be shown to be a lattice. [4] We will use the notations

$$\begin{cases} 0_n &= \{\{1\}, \{2\}, \ldots, \{n\}\} \\ 1_n &= \{\ \{1, 2, \ldots, n\}\ \} \end{cases} \tag{2.1}$$

for the minimal and maximal element of $NC(n)$, respectively.

The lattice of non-crossing partitions was introduced by G. Kreweras in [4], and its combinatorics has been studied by several authors (see e.g. [7], and the list of references there). We will only insist here on one basic concept (the Kreweras complementation map), which will be extensively used in the Sections 3 and 4 of the paper.

**2.2 The complementation map of Kreweras** is a remarkable lattice anti-isomorphism $K : NC(n) \to NC(n)$, introduced in [4], Section 3, and described as follows.

We will use the *circular representation* of a partition $\pi = \{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$, which consists in drawing $n$ equidistant and clockwisely ordered points $P_1, \ldots, P_n$ on a circle, and in drawing, for each block $B_j$ ($1 \leq j \leq k$) of $\pi$, the convex hull $H_j$ of the points $\{P_m \mid m \in B_j\}$. It is easily verified that $\pi$ is non-crossing if and only if the $k$ convex hulls $H_1, \ldots, H_k$ are pairwisely disjoint. Moreover, if $\pi$ is non-crossing, then - denoting by $D$ the convex hull of the whole circle - it is also easily verified that $D \setminus (H_1 \cup \cdots \cup H_k)$ has exactly $n+1-k$ connected components $\widetilde{H}_1, \ldots, \widetilde{H}_{n+1-k}$, each of them being itself a convex set. (In order to count the connected components of $D \setminus (H_1 \cup \cdots \cup H_k)$, one can proceed by drawing $H_1, \ldots, H_k$ one by one. First, $D \setminus H_1$ has $|B_1|$ connected components, then the drawing of each $H_j$, $2 \leq j \leq k$, increases the number of connected components of the complement by $|B_j| - 1$; so in the end the complement has $|B_1| + (|B_2| - 1) + \cdots + (|B_k| - 1)$ $= (|B_1| + \cdots + |B_k|) - (k-1) = n + 1 - k$ components.)

Now, let $\pi = \{B_1, \ldots, B_k\}$ be in $NC(n)$, and consider the points $P_1, \ldots, P_n$ and the decomposition $D = (H_1 \cup \cdots \cup H_k) \cup (\widetilde{H}_1 \cup \cdots \cup \widetilde{H}_{n+1-k})$, as described in the preceding paragraph. Denote the midpoints of the arcs $P_1P_2, \ldots, P_{n-1}P_n,\ P_nP_1$ by $Q_1, \ldots, Q_n$, respectively. Then:

[4] This means that every two elements $\pi, \rho \in NC(n)$ have a lowest upper bound $\pi \vee \rho \in NC(n)$, and a greatest lower bound $\pi \wedge \rho \in NC(n)$. The operations "$\vee$" and "$\wedge$" will not be explicitly used in what follows; thus, when we speak of a "lattice isomorphism" (or "lattice anti-isomorphism"), this may be taken just as an isomorphism (respectively anti-isomorphism) for the order structure.

**Definition** The *Kreweras complement* of $\pi$, denoted by $K(\pi)$, is the partition of $\{1,\dots,n\}$ determined by

$$i \overset{K(\pi)}{\sim} j \overset{def}{\Leftrightarrow} \left\{ \begin{array}{c} Q_i \text{ and } Q_j \text{ belong to the same convex} \\ \text{set } \widetilde{H}_l\text{,for some } 1 \leq l \leq n+1-k \end{array} \right\} \tag{2.2}$$

$K(\pi)$ is non-crossing , as it is obvious by looking at its circular representation based on the points $Q_1,\dots,Q_n$; note also that $K(\pi)$ has exactly $n+1-k$ blocks (because, as it is easy to see, each $\widetilde{H}_l$, $1 \leq l \leq n+1-k$, must contain at least one point $Q_m$).

As a concrete example, Figure 1 illustrates that $K(\{\{1,4,8\},\{2,3\}, \{5,6\},\{7\}\}) = \{\{1,3\},\{2\},\{4,6,7\},\{5\},\{8\}\} \in NC(8)$.

The following statement is a mere reformulation of the definition of the Kreweras complement, and its proof is left as an exercise.

**Proposition** Let $\pi$ and $\rho$ be in $NC(n)$. Denote by $\pi'$ and $\rho'$ the partitions of $\{2,4,6,\dots, 2n\}$ and $\{1,3,5,\dots,2n-1\}$, respectively, which get identified to $\pi$ and $\rho$ via the order-preserving bijections $\{1,\dots,n\} \to \{2,4,6,\dots,2n\}$ and $\{1,\dots,n\} \to \{1,3,5,\dots,2n-1\}$. Denote by $\sigma$ the partition of $\{1,2,3,\dots,2n\}$ formed by $\pi'$ and $\rho'$ together. Then $\sigma$ is non-crossing if and only if $\pi \leq K(\rho)$.

The fact, mentioned at the beginning of this subsection, that $K : NC(n) \to NC(n)$ is an anti-isomorphism, for every $n \geq 1$, can be proved for instance in the following way:

first, it is immediately seen that $K^2(\pi)$ is (for every $\pi \in NC(n)$) the anti-clockwise rotation of $\pi$ with $360^o/n$, and this shows in particular that $K$ is a bijection; then, it is also easy to check that $\pi \leq \rho \Rightarrow K(\pi) \geq K(\rho)$ - and the converse must also hold, since $K^2$ is an order-preserving isomorphism of $NC(n)$.

**2.3 A relation with the permutation group** Let $\mathcal{S}_n$ denote the group of all permutations of $\{1, \ldots, n\}$. For $B = \{i_1, i_2, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, we denote by $\tau_B \in \mathcal{S}_n$ the cycle given by

$$\begin{cases} \tau_B(i_1) = i_2, \ldots, \tau_B(i_{m-1}) = i_m, \tau_B(i_m) = i_1, \\ \tau_B(j) = j \text{ for } j \in \{1, \ldots, n\} \setminus B \end{cases} \tag{2.3}$$

(if $|B| = 1$, then $\tau_B$ is of course the unit element of $\mathcal{S}_n$; at the other end, $\tau_{\{1,\ldots,n\}}$ is the cycle $(1 \to 2 \to \cdots \to n \to 1)$ ).

We will make use of a remarkable injective map from $NC(n)$ into $\mathcal{S}_n$, which is denoted here by $Perm$, and is described as follows:

**Definition** Let $\pi = \{B_1, \ldots, B_k\}$ be a non-crossing partition of $\{1, \ldots, n\}$. We denote by $Perm(\pi) \in \mathcal{S}_n$ the permutation which has cycle decomposition: $Perm(\pi) = \tau_{B_1} \cdots \tau_{B_k}$, with $\tau_{B_j}$, $1 \leq j \leq k$, as defined in (2.3).

The embedding $Perm : NC(n) \to \mathcal{S}_n$ was introduced and studied by Ph. Biane in [2], where $Perm(\pi)$ is referred to as "the trace of the cycle $(1 \to 2 \to \cdots \to n \to 1)$ on the partition $\pi$." The result from [2] which will be needed here is that the Kreweras complementation map has a nice interpretation in this context, namely we have ([2], Section 1.4.2):

$$Perm(\pi) \cdot Perm(K(\pi)) \;=\; Perm(1_n) \;(=\; (1 \to 2 \to \cdots \to n \to 1) \;), \tag{2.4}$$

for every $\pi \in NC(n)$.

**2.4 The relative Kreweras complement** Let $\rho$ be a fixed non-crossing partition of $\{1, \ldots, n\}$. We will need a version of the Kreweras complementation map which is considered "relatively to $\rho$", i.e. it is an anti-automorphism $K_\rho$ of the sublattice $\{\pi \in NC(n) \mid \pi \leq \rho\}$ of $NC(n)$ (the usual Kreweras complementation will correspond to the case $\rho = 1_n$). Roughly speaking, the relative complement $K_\rho(\pi)$ of $\pi \leq \rho$ is obtained by looking at how $\pi$ splits

the blocks of $\rho$, and then taking the usual Kreweras complement inside each block. A more formal definition can be made as follows.

Remark first that we have a natural lattice-isomorphism

$$\{\pi \in NC(n) \mid \pi \leq \rho\} \xrightarrow{\sim} NC(|B_1|) \times \cdots \times NC(|B_k|), \tag{2.5}$$

where $B_1, \ldots, B_k$ are the blocks of $\rho$ (and $|B_j|$ denotes the number of elements of $B_j$). In order to describe the image of a given $\pi \leq \rho$ under the map (2.5), one writes the blocks of $\pi$ as $A_{1,1}, \ldots, A_{1,p_1}, \ldots, A_{k,1}, \ldots, A_{k,p_k}$, such that $A_{j,1} \cup \cdots \cup A_{j,p_j} = B_j$ for $1 \leq j \leq k$. Now, if one relabels the elements of $B_j$ (taken in increasing order) as $1, 2, \ldots, |B_j|$, then the partition $\{A_{j,1}, \ldots, A_{j,p_j}\}$ of $B_j$ gets relabeled into a partition $\pi_j \in NC(|B_j|),\ 1 \leq j \leq k$; the image of $\pi$ under (2.5) is, by definition, the $k$-tuple $(\pi_1, \ldots, \pi_k)$. It is immediate that (2.5) really is a bijection, and in fact a lattice-isomorphism.

Then we can write:

**Definition** The relative Kreweras complementation map $K_\rho$ is the unique map closing the square

$$\begin{array}{ccc} \{\pi \in NC(n) \mid \pi \leq \rho\} & \overset{\sim}{\dashrightarrow} & NC(|B_1|) \times \cdots \times NC(|B_k|) \\ \Big\downarrow K_\rho & & \Big\downarrow K \times \cdots \times K \\ \{\pi \in NC(n) \mid \pi \leq \rho\} & \overset{\sim}{\dashrightarrow} & NC(|B_1|) \times \cdots \times NC(|B_k|) \end{array} \tag{2.6}$$

where on the horizontals we have the isomorphism of (2.5), and on the right vertical we have the direct product of the (usual) complementation maps on $NC(|B_1|), \ldots, NC(|B_k|)$.

**2.5 Relation with the permutation groups (continued)** The Equation (2.4) mentioned in Section 2.3 is also having a relativized analogue, namely

$$Perm(\pi) \cdot Perm(K_\rho(\pi)) \ = \ Perm(\rho), \tag{2.7}$$

for every $\pi, \rho \in NC(n)$ such that $\pi \leq \rho$. The verification of (2.7) is easily done by using (2.4) and (2.6), and is left to the reader.

The following partial converse to (2.7) (or rather the Corollary following to it) will be used in Section 3.5.

**Proposition** Let $\widetilde{\pi}, \widetilde{\sigma}, \widetilde{\rho}$ be non-crossing partitions of $\{1, \ldots, n\}$, such that $\widetilde{\sigma} \leq \widetilde{\rho}$, and

$$Perm(\widetilde{\pi}) \cdot Perm(\widetilde{\sigma}) \ = \ Perm(\widetilde{\rho}). \tag{2.8}$$

Then we also have that $\widetilde{\pi} \leq \widetilde{\rho}$, and moreover, that $K_{\widetilde{\rho}}(\widetilde{\pi}) = \widetilde{\sigma}$.

**Proof** The blocks of $\widetilde{\pi}$ are recaptured as the orbits of $\{1, \ldots, n\}$ under the action of the permutation $Perm(\widetilde{\pi})$. On the other hand, the blocks of $\widetilde{\rho}$ are invariant under the action of both $Perm(\widetilde{\rho})$ and $Perm(\widetilde{\sigma})$ (where the latter assertion follows from the hypothesis that $\widetilde{\sigma} \leq \widetilde{\rho}$); hence the blocks of $\widetilde{\rho}$ must also be invariant for $Perm(\widetilde{\rho}) \cdot (Perm(\widetilde{\sigma}))^{-1} \overset{(2.8)}{=} Perm(\widetilde{\pi})$. This clearly entails that every block of $\widetilde{\rho}$ is a union of blocks of $\widetilde{\pi}$, i.e. that $\widetilde{\pi} \leq \widetilde{\rho}$. Finally, by comparing (2.7) with (2.8) we obtain that $Perm(\widetilde{\sigma}) = Perm(K_{\widetilde{\rho}}(\widetilde{\pi}))$, hence that $\widetilde{\sigma} = K_{\widetilde{\rho}}(\widetilde{\pi})$. **QED**

**Corollary** Let $\pi, \rho$ be non-crossing partitions of $\{1, \ldots, n\}$, such that $\pi \leq \rho$. Then we have that $K_{\rho}(\pi) \leq K(\pi)$, and moreover, that

$$K_{K(\pi)}(K_{\rho}(\pi)) \ = \ K(\rho). \tag{2.9}$$

**Proof** It suffices to verify the equality

$$Perm(K_{\rho}(\pi)) \cdot Perm(K(\rho)) \ = \ Perm(K(\pi)), \tag{2.10}$$

because we also know that $K(\rho) \leq K(\pi)$ (as implied by the hypothesis $\pi \leq \rho$, and the fact that $K(\cdot)$ is order-reversing), so after that we will just have to apply the preceding Proposition, with $\widetilde{\pi} = K_{\rho}(\pi)$, $\widetilde{\sigma} = K(\rho)$, $\widetilde{\rho} = K(\pi)$. But(2.10) follows immediately if we replace $Perm(K(\rho))$ and $Perm(K(\pi))$ from (2.4), and $Perm(K_{\rho}(\pi))$ from (2.7). **QED**

**2.6 Non-crossing pairings** A partition $\pi$ of $\{1, 2, \ldots, 2n\}$ is called a *pairing* if every block of $\pi$ has exactly two elements. We denote the set of all non-crossing pairings of $\{1, 2, \ldots, 2n\}$ by $NCP(2n)$. Non-crossing pairings have been studied for quite some time before non-crossing partitions were introduced, see e.g. [11]; it is interesting that they are also counted by the Catalan numbers, $|NCP(2n)| = (2n)!/(n!(n+1)!)$ for $n \geq 1$.

In particular, we have $|NC(n)| = |NCP(2n)|$ for every $n \geq 1$. This is not a coincidence, and the following "canonical" bijection between $NC(n)$ and $NCP(2n)$ will be needed in the proof of the Application 1.10.

In the next Proposition, in order to avoid any possibility of confusion, we write $K_{(n)}$ and $K_{(2n)}$ for the Kreweras complementation maps on $NC(n)$ and $NC(2n)$, respectively.

**Proposition** Let $n$ be a positive integer. For every $\rho \in NC(n)$, we denote by $Twice(\rho)$ the partition of $\{1, 2, \ldots, 2n\}$ which is obtained by "letting $\rho$ work on $\{1, 3, \ldots, 2n-1\}$ and letting $K_{(n)}(\rho)$ work on $\{2, 4, \ldots, 2n\}$", in the same sense as in the Proposition in 2.2. Then $Twice(\rho)$ is in $NC(2n)$, and the map

$$NC(n) \ni \rho \longrightarrow K_{(2n)}(Twice(\rho)) \in NC(2n) \tag{2.11}$$

is a bijection from $NC(n)$ onto $NCP(2n)$.

**Proof** $Twice(\rho)$ is non-crossing by the Proposition in 2.2. The map (2.11) is obviously one-to-one from $NC(n)$ into $NC(2n)$, so we only need to verify that it takes values in $NCP(2n)$ (a one-to-one map from $NC(n)$ into $NCP(2n)$ is necessarily onto, since $|NC(n)| = |NCP(2n)|$).

So, let us fix $\rho \in NC(n)$, and let us show that $K_{(2n)}(Twice(\rho)) \in NCP(2n)$. The simplest way of doing this is probably via the connection with the permutation groups discussed in 2.3, 2.5. We denote $Perm(\rho) \in \mathcal{S}_n$ by $\xi$, and let us also use the notations $\gamma_n, \gamma_{2n}$ for the cycles $Perm(1_n) = (1 \to 2 \to \cdots \to n \to 1) \in \mathcal{S}_n$ and $Perm(1_{2n}) = (1 \to 2 \to \cdots \to 2n \to 1) \in \mathcal{S}_{2n}$, respectively. Then $Perm(K_{(n)}(\rho)) = \xi^{-1}\gamma_n$ by (2.4), and it is immediate that $Perm(Twice(\rho)) \in \mathcal{S}_{2n}$ is acting by

$$\begin{cases} 2k-1 & \to & 2\xi(k)-1, & 1 \leq k \leq n; \\ 2k & \to & 2\xi^{-1}(k+1), & 1 \leq k < n; \\ 2n & \to & 2\xi^{-1}(1). & \end{cases} \tag{2.12}$$

Now, $Perm(K_{(2n)}(Twice(\rho))) = Perm(Twice(\rho))^{-1} \cdot \gamma_{2n}$, also by (2.4). Taking (2.12) into account, we obtain that $Perm(K_{(2n)}(Twice(\rho)))^{-1} = \gamma_{2n}^{-1} \cdot Perm(Twice(\rho))$ acts by:

$$\begin{cases} 2k-1 & \to & 2\xi(k)-2, & \text{for } 1 \leq k \leq n, \ k \neq \xi^{-1}(1); \\ 2k-1 & \to & 2n, & \text{if } k = \xi^{-1}(1); \\ 2k & \to & 2\xi^{-1}(k+1)-1, & 1 \leq k < n; \\ 2n & \to & 2\xi^{-1}(1)-1. & \end{cases} \tag{2.13}$$

As one can check without difficulty, (2.13) shows that $Perm(K_{(2n)}(Twice(\rho)))^{-1}$ is a product of $n$ disjoint transpositions. Hence the same holds for $Perm(K_{(2n)}(Twice(\rho)))$ itself, and this means exactly that $K_{(2n)}(Twice(\rho)) \in NCP(2n)$. **QED**

## 3. The operation $\boxed{\star}$ on formal power series

**3.1 Notations** Let $n$ be a positive integer. Recall from Section 1.2 that we denote by $\Theta_n$ the set of formal power series without constant coefficient in $n$ non-commuting variables $z_1, \ldots, z_n$ (i.e., series of the form appearing in Eqn.(1.3) above). For $f \in \Theta_n$ and $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$, we will denote

$$[\text{coef } (i_1, \ldots, i_k)](f) \overset{def}{=} \text{the coefficient of } z_{i_1} \cdots z_{i_k} \text{ in } f. \tag{3.1}$$

The following conventions for denoting coefficients will also turn out to be handy in what follows.

(a) Let $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$ be integers, and let $B = \{b_1, b_2, \ldots, b_l\}$ be a non-void subset of $\{1, \ldots, k\}$, where $1 \leq b_1 < b_2 < \cdots < b_l \leq k$. Then by "$(i_1, \ldots, i_k)|B$" we will understand the $l$-tuple $(i_{b_1}, i_{b_2}, \ldots, i_{b_l})$. (E.g., $(i_1, i_2, i_3, i_4, i_5)|\{2, 3, 5\} = (i_2, i_3, i_5)$.)

(b) Let $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$ be integers, and let $\pi$ be a partition of $\{1, \ldots, k\}$. Then for every $f \in \Theta_n$ we put

$$[\text{coef } (i_1, \ldots, i_k); \pi](f) \overset{def}{=} \prod_{B \text{ block of } \pi} [\text{coef } (i_1, \ldots, i_k)|B](f). \tag{3.2}$$

(Thus, for example, if $k = 4$ and $\pi = \{\{1, 3\}, \{2\}, \{4\}\}$, then

$$[\text{coef } (i_1, i_2, i_3, i_4); \pi](f) = [\text{coef } (i_1, i_3)](f) \cdot [\text{coef } (i_2)](f) \cdot [\text{coef } (i_4)](f),$$

for every $f \in \Theta_n$ and $1 \leq i_1, i_2, i_3, i_4 \leq n$.)

Note that if $\pi$ in (3.2) is the partition (denoted in (2.1) by $1_k$) of $\{1, \ldots, k\}$ into only one block, then $[\text{coef } (i_1, \ldots, i_k); \pi](f)$ is just the coefficient $[\text{coef } (i_1, \ldots, i_k)](f)$ appearing in (3.1). It is clear that (given $k$ and $i_1, \ldots, i_k$) this is the only choice of $\pi$ for which the functional $[\text{coef } (i_1, \ldots, i_k); \pi] : \Theta_n \to \mathbf{C}$ is linear.

**3.2 Definition** Let $n$ be a positive integer. We denote by $\boxed{\star}$ the binary operation on the set $\Theta_n$ of 3.1, which is determined by the equation

$$[\text{coef } (i_1, \ldots, i_k)](f \boxed{\star} g) = \sum_{\pi \in NC(k)} [\text{coef } (i_1, \ldots, i_k); \pi](f) \cdot [\text{coef } (i_1, \ldots, i_k); K(\pi)](g), \tag{3.3}$$

holding for every $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$, and where $K : NC(k) \to NC(k)$ is the Kreweras complementation map reviewed in Section 2.2.

**3.3 Remark** While (3.3) determines $f \boxed{\star} g$ completely, it is useful to note that it can be extended to:

$$[\text{coef } (i_1, \ldots, i_k); \rho](f \boxed{\star} g) \;=\; \sum_{\substack{\pi \in NC(k) \\ \pi \leq \rho}} [\text{coef } (i_1, \ldots, i_k); \pi](f) \cdot [\text{coef } (i_1, \ldots, i_k); K_\rho(\pi)](g), \tag{3.4}$$

holding for $k \geq 1,\ 1 \leq i_1, \ldots, i_k \leq n$, and some arbitrary partition $\rho \in NC(k)$. Eqn.(3.4) is obtained in a straightforward way from (3.3) and the considerations in Section 2.4.

**3.4 Remark** If in the preceding definition we take $n = 1$, then the operation $\boxed{\star}$ (on $\Theta_1$) has a definite combinatorial significance, and was analyzed in [8], [6] in an approach where it is called "the convolution of multiplicative functions in the large incidence algebra on non-crossing partitions". It would be possible to adapt this approach to the multidimensional situation, and place our considerations in the framework of what is called "the Moebius inversion theory" of a certain partially ordered set of "colored non-crossing partitions". However, since the properties of $\boxed{\star}$ that are needed here can be derived in a self-contained and elementary way, we have chosen not to enter into any details in this direction (the reader interested in having a look at multiplicative functions on non-crossing partitions is referred to [8], Section 3, or [6], Section 1; for the general theory of Moebius inversion on partially ordered sets, see e.g. [10], Chapter 3).

**3.5 Proposition** Let $n$ be a positive integer. The binary operation $\boxed{\star}$ on $\Theta_n$ defined in 3.2 is associative.

**Proof** Consider (and fix) the series $f, g, h \in \Theta_n$ and the numbers $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$, about which we want to show that

$$[\text{coef } (i_1, \ldots, i_k)]((f \boxed{\star} g) \boxed{\star} h) \;=\; [\text{coef } (i_1, \ldots, i_k)](f \boxed{\star} (g \boxed{\star} h)). \tag{3.5}$$

We have:

$$[\text{coef } (i_1, \ldots, i_k)]((f \boxed{\star} g) \boxed{\star} h) \overset{(3.3)}{=}$$

$$= \sum_{\rho \in NC(k)} [\text{coef } (i_1, \ldots, i_k); \rho](f \boxed{\star} g) \cdot [\text{coef } (i_1, \ldots, i_k); K(\rho)](h) \overset{(3.4)}{=}$$

$$= \sum_{\substack{\pi \leq \rho \\ in\ NC(k)}} [\text{coef } (i_1,\ldots,i_k);\pi](f)\cdot[\text{coef } (i_1,\ldots,i_k);K_\rho(\pi)](g)\cdot[\text{coef } (i_1,\ldots,i_k);K(\rho)](h). \tag{3.6}$$

In a similar way, we see that

$$[\text{coef } (i_1,\ldots,i_k)](f \boxed{\star} (g \boxed{\star} h)) \stackrel{(3.3)}{=}$$

$$= \sum_{\pi \in NC(k)} [\text{coef } (i_1,\ldots,i_k);\pi](f)\cdot[\text{coef } (i_1,\ldots,i_k);K(\pi)](g \boxed{\star} h) \ =$$

(via the substitution $K(\pi) = \sigma$)

$$= \sum_{\sigma \in NC(k)} [\text{coef } (i_1,\ldots,i_k);K^{-1}(\sigma)](f)\cdot[\text{coef } (i_1,\ldots,i_k);\sigma](g \boxed{\star} h) \stackrel{(3.4)}{=}$$

$$\sum_{\substack{\tau \leq \sigma \\ in\ NC(k)}} [\text{coef } (i_1,\ldots,i_k);K^{-1}(\sigma)](f)\cdot[\text{coef } (i_1,\ldots,i_k);\tau](g)\cdot[\text{coef } (i_1,\ldots,i_k);K_\sigma(\tau)](h). \tag{3.7}$$

We are left to establish the equality of the sums in (3.6) and (3.7). We will do this by showing that: the map

$$(\pi,\rho) \ \longrightarrow \ (\tau,\sigma) \stackrel{def}{=} (K_\rho(\pi), K(\pi)) \tag{3.8}$$

is a bijection from $\{(\pi,\rho) \mid \pi,\rho \in NC(k),\ \pi \leq \rho\}$ onto itself, which identifies the sums (3.6) and (3.7) term by term.

The fact that the map (3.8) really takes its domain into itself follows from Corollary 2.5. In order to show that (3.8) is a bijection, it suffices to check injectivity; and indeed, from $(K_\rho(\pi), K(\pi)) = (K_{\rho'}(\pi'), K(\pi'))$ we get first that $K(\pi) = K(\pi') \Rightarrow \pi = \pi'$, and then (in the notations of 2.5):

$$K_\rho(\pi) = K_{\rho'}(\pi) \ \stackrel{(2.7)}{\Rightarrow} \ (Perm(\pi))^{-1}(Perm(\rho)) = (Perm(\pi))^{-1}(Perm(\rho'))$$

$$\Rightarrow \ Perm(\rho) = Perm(\rho') \ \Rightarrow \ \rho = \rho'.$$

Finally, let us verify that for every $\pi \leq \rho$ in $NC(k)$, the term in (3.6) corresponding to $(\pi,\rho)$ is equal to the term in (3.7) corresponding to $(\tau,\sigma) \stackrel{def}{=} (K_\rho(\pi), K(\pi))$; this comes, clearly, to verifying that $\pi = K^{-1}(\sigma)$, $K_\rho(\pi) = \tau$, $K(\rho) = K_\sigma(\tau)$. And indeed, the first two of the latter equalities are obvious, while the third one coincides with (2.9) of Corollary 2.5.
**QED**

The argument proving the next proposition is very similar to the one used to establish the Moebius inversion formula in a partially ordered set (compare for instance to [10], Proposition 3.6.2); we will skip here the details of the straightforward, but rather space-consuming proof of the $2^o$"$\Leftarrow$" part.

**3.6 Proposition** Let $n$ be a positive integer.

$1^o$ The binary operation $\boxed{\star}$ on $\Theta_n$ has a neutral element, which is the series

$$Sum(z_1, \ldots , z_n) \stackrel{def}{=} z_1 + \cdots + z_n. \tag{3.9}$$

$2^o$ A series $f \in \Theta_n$ is invertible with respect to $\boxed{\star}$ if and only if its coefficients of degree one, $[\text{coef } (i)](f)$, $1 \leq i \leq n$, are all non-zero.

**Proof** $1^o$ is immediate.

$2^o$"$\Rightarrow$": If $g$ denotes the inverse of $f$ under $\boxed{\star}$, then for every $1 \leq i \leq n$ we have:

$$1 = [\text{coef } (i)](Sum) \ = \ [\text{coef } (i)](f \boxed{\star} g) \stackrel{(3.3)}{=} [\text{coef } (i)](f) \cdot [\text{coef } (i)](g),$$

which implies that $[\text{coef } (i)](f) \neq 0$.

$2^o$"$\Leftarrow$": Say that we want to define a family of complex numbers, $(\beta_{(i_1,\ldots,i_k)})_{k\geq 1, 1\leq i_1,\ldots,i_k \leq n}$, such that the series

$$g(z_1, \ldots , z_n) \stackrel{def}{=} \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} \beta_{(i_1,\ldots,i_k)} z_{i_1} \cdots z_{i_k}$$

has the property that $g \boxed{\star} f \ = \ Sum$. The identification of the coefficients in the latter equality comes to an (infinite) system of equations, and each of these equations involves (by (3.3)) a summation over a lattice of non-crossing partitions $NC(k)$. If one separates in each such summation over $NC(k)$ the term corresponding to the partition with only one block, $\{ \{1, 2, \ldots , k\} \}$, then one sees without difficulty that in fact the infinite system of equations considered does nothing else but defining the desired coefficients $(\beta_{(i_1,\ldots,i_k)})_{k\geq 1, 1\leq i_1,\ldots,i_k \leq n}$, by induction on $k$; this implies, in other words, that $f$ has a (unique) inverse on the left under $\boxed{\star}$. The existence of an inverse on the right is shown in a similar manner, and then, as it is well-known, the two inverses must coincide because of the associativity of $\boxed{\star}$. **QED**

We now turn towards giving the precise definition of the multivariable $R$-transform, which was deferred from the Section 1.2. On the line taken here, it is convenient to define

the $R$-transform in terms of the operation $\boxed{\star}$ of 3.2 (though, of course, this doesn't correspond to the chronological development).

We need to introduce first the version that is appropriate, in the present framework, for the zeta and Moebius function from the Moebius inversion theory in partially ordered sets (compare for instance to [10], Section 3.7).

**3.7 Definition** Let $n$ be a positive integer. We will call *zeta power series in $n$ variables,* and denote by $Zeta$ (or $Zeta_n$, if the precisation of $n$ is needed), the element of $\Theta_n$ given by

$$Zeta(z_1, \ldots , z_n) \;=\; \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} z_{i_1} \cdots z_{i_k}. \tag{3.10}$$

$Zeta$ is invertible in $(\Theta_n, \boxed{\star})$, by Proposition 3.6; its inverse will be called the *Moebius series in $n$ variables,* and will be denoted by $Moeb$ (or $Moeb_n$).

**3.8 Remark** It is not hard to write down the Moebius series explicitly, the formula is:

$$Moeb(z_1, \ldots , z_n) \;=\; \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} (-1)^{k+1} \frac{(2k-2)!}{(k-1)!k!} \, z_{i_1} \cdots z_{i_k} \tag{3.11}$$

(note again the occurrence of the Catalan numbers). The shortest way for deriving (3.11) goes probably by noticing that its verification doesn't depend on $n$, and then by invoking the literature existent in the case $n = 1$, when $Moeb$ really is the Moebius series associated to the lattices of non-crossing partitions (see, e.g., the Corollary 5 in Section 3 of [8]).

**3.9 Definition** Let $n$ be a positive integer. Let $\Theta_n$ be as above, and consider, as in Section 1.2, the set of linear functionals $\Sigma_n = \{\mu : \mathbf{C}\langle X_1, \ldots , X_n\rangle \to \mathbf{C} \mid \mu \text{ linear}, \mu(1) = 1\}$. For every $\mu \in \Sigma_n$ we denote, as in Section 1.5, by $M(\mu) \in \Theta_n$ the formal power series which has the moments of $\mu$ as coefficients. Then the $n$-dimensional $R$-transform is the bijection $R : \Sigma_n \to \Theta_n$ defined by the formula

$$R(\mu) \;=\; M(\mu)\boxed{\star}Moeb, \quad \mu \in \Sigma_n. \tag{3.12}$$

Note that an equivalent way of writing Eqn.(3.12) is

$$M(\mu) \;=\; R(\mu)\boxed{\star}Zeta, \quad \mu \in \Sigma_n; \tag{3.13}$$

(3.13) is in some sense "the formula for the $R^{-1}$-transform", since the transition from $M(\mu)$ back to $\mu$ is trivial.

Following [8], we will also refer to the coefficients of $R(\mu)$ under the name of *free cumulants* of $\mu$. The above Equations (3.13), (3.12) are in fact just a way of rewriting the version of the Equations ($\star$) and ($\star\star$) in [8, Section 4] which applies to the framework considered here.

We also mention that an alternative description of the $n$-dimensional $R$-transform, made in terms of "creation and annihilation operators on the full Fock space over $\mathbf{C}^n$" is presented in [5]; this makes the connection between the definition given above and the original approach of Voiculescu in [13].

We are only left now to present the proof of the formula (1.12) in Theorem 1.4. Before doing this, we would like to make the following important

**3.10 Remark:** Equation (1.12) in Theorem 1.4 is equivalent to:

$$M(\mu_{a_1b_1,\ldots,a_nb_n}) \;=\; R(\mu_{a_1,\ldots,a_n}) \boxed{\star} M(\mu_{b_1,\ldots,b_n}). \tag{3.14}$$

Indeed, (3.14) is obtained from (1.12) by $\boxed{\star}$-operating with $Zeta$ on the right, and the converse transition is performed by $\boxed{\star}$-operating on the right with $Moeb$.

We take the occasion to note here that $Zeta$ and $Moeb$ *lie in the centre* of the semigroup $(\Theta_n, \boxed{\star}\,)$; this is immediately verified for $Zeta$ by using the formula (3.3) and the bijectivity of the Kreweras complementation map (and then, of course, it also follows for $Moeb = Zeta^{-1}$). As a consequence, when $\boxed{\star}$-operating with $Zeta$ on the right in Eqn.(1.12), we can also associate the factor $Zeta$ (on the right-hand side) to $R(\mu_{a_1,\ldots,a_n})$, and thus bring (1.12) to another equivalent form,

$$M(\mu_{a_1b_1,\ldots,a_nb_n}) \;=\; M(\mu_{a_1,\ldots,a_n}) \boxed{\star} R(\mu_{b_1,\ldots,b_n}). \tag{3.15}$$

**3.11 The proof of Theorem 1.4** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ be such that $\{a_1, \ldots, a_n\}$ is free from $\{b_1, \ldots, b_n\}$. We will prove that the equality (3.14) (equivalent, as noted above, to (1.12) of 1.4) takes place. Thus, we fix $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$, and we will show that the coefficients of $z_{i_1} \cdots z_{i_k}$ in $M(\mu_{a_1b_1,\ldots,a_nb_n})$ and $R(\mu_{a_1,\ldots,a_n}) \boxed{\star} M(\mu_{b_1,\ldots,b_n})$ are equal. The line of proof is the same as in [6], Section 3.4 (see also [9], Section 3.4).

We have:

$$[\text{coef } (i_1, \ldots, i_k)](M(\mu_{a_1b_1,\ldots,a_nb_n})) \;=\; \varphi(a_{i_1}b_{i_1} \cdots a_{i_k}b_{i_k}) \;=$$

$$= [\text{coef } (i_1, i_1 + n, \ldots, i_k, i_k + n)](M(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n})), \tag{3.16}$$

where $M(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n}) \in \Theta_{2n}$ will be viewed as acting in the $2n$ variables $z_1, \ldots, z_{2n}$, with $z_1, \ldots, z_n$ corresponding to the $a$'s and $z_{n+1}, \ldots, z_{2n}$ corresponding to the $b$'s. But we know that

$$M(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n}) \overset{(3.13)}{=} R(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n}) \boxed{\star} Zeta_{2n},$$

so (according to Eqn.(3.3) in 3.2 and the definition of $Zeta$), (3.16) can be continued with:

$$\sum_{\sigma \in NC(2k)} [\text{coef } (i_1, i_1 + n, \ldots, i_k, i_k + n); \sigma] \left( R(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n}) \right). \tag{3.17}$$

Now, the hypothesis that $a_1, \ldots, a_n$ is free from $b_1, \ldots, b_n$ has the consequence that

$$[R(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n})](z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}) = \tag{3.18}$$

$$[R(\mu_{a_1,\ldots,a_n})](z_1, \ldots, z_n) + [R(\mu_{b_1,\ldots,b_n})](z_{n+1}, \ldots, z_{2n})$$

(see Eqn.(1.4) above); this makes clear that a partition $\sigma \in NC(2k)$ can bring a non-zero contribution to the sum (3.17) only if each block of $\sigma$ either is contained in $\{1, 3, \ldots, 2k - 1\}$, or is contained in $\{2, 4, \ldots, 2k\}$. But from the Proposition in Section 2.2 it follows immediately that the set of partitions $\sigma \in NC(2k)$ having the latter property is in natural bijection with the set $\{(\pi, \rho) \mid \pi, \rho \in NC(k), \ \pi \leq K(\rho)\}$, in the following way: given $\pi, \rho \in NC(k)$, with $\pi \leq K(\rho)$, the partition $\sigma \in NC(2k)$ corresponding to $(\pi, \rho)$ is obtained by "letting $\pi$ work on $\{2, 4, \ldots, 2k\}$ and letting $\rho$ work on $\{1, 3, \ldots, 2k - 1\}$". From the description of the bijection and from (3.18) it is immediate that, whenever $\pi, \rho, \sigma$ are as in the preceding phrase, we have

$$[\text{coef } (i_1, i_1 + n, \ldots, i_k, i_k + n); \sigma] \left( R(\mu_{a_1,\ldots,a_n,b_1,\ldots,b_n}) \right) =$$

$$[\text{coef } (i_1, \ldots, i_k); \rho] \left( R(\mu_{a_1,\ldots,a_n}) \right) \cdot [\text{coef } (i_1, \ldots, i_k); \pi] \left( R(\mu_{b_1,\ldots,b_n}) \right). \tag{3.19}$$

By putting together (3.16), (3.17), (3.19), we obtain that:

$$[\text{coef } (i_1, \ldots, i_k)](M(\mu_{a_1 b_1,\ldots,a_n b_n})) =$$

$$= \sum_{\substack{\pi,\rho \text{ in } NC(k) \\ \text{such that} \\ \pi \leq K(\rho)}} [\text{coef } (i_1, \ldots, i_k); \rho](R(\mu_{a_1,\ldots,a_n})) \cdot [\text{coef } (i_1, \ldots, i_k); \pi](R(\mu_{b_1,\ldots,b_n}))$$

$$= \sum_{\rho \in NC(k)} [\text{coef } (i_1, \dots, i_k); \rho](R(\mu_{a_1,\dots,a_n})) \cdot ( \sum_{\substack{\pi \leq K(\rho) \\ in \ NC(k)}} [\text{coef } (i_1, \dots, i_k); \pi](R(\mu_{b_1,\dots,b_n})) \ ). \tag{3.20}$$

Finally, by using Eqn.(3.4) we infer that the second sum in (3.20) is equal to $[\text{coef } (i_1, \dots, i_k); K(\rho)](R(\mu_{b_1,\dots,b_n}) \boxed{\star} Zeta)$; and since we know that $R(\mu_{b_1,\dots,b_n}) \boxed{\star} Zeta$ is the same thing as $M(\mu_{b_1,\dots,b_n})$, we can conclude that the quantity in (3.20) equals

$$\sum_{\rho \in NC(k)} [\text{coef } (i_1, \dots, i_k); \rho](R(\mu_{a_1,\dots,a_n})) \cdot [\text{coef } (i_1, \dots, i_k); K(\rho)](M(\mu_{b_1,\dots,b_n}))$$

$$\stackrel{(3.3)}{=} \ [\text{coef } (i_1, \dots, i_k)] \left( R(\mu_{a_1,\dots,a_n}) \boxed{\star} M(\mu_{b_1,\dots,b_n}) \right),$$

as desired. **QED**

**3.12 Remark** As we recalled in Section 1.3, the 1-dimensional instance of Theorem 1.4 is related to the $S$-transform of Voiculescu, and it is natural to ask whether (and to what extent) could the $S$-transform itself be adapted to work in the multidimensional case. We are not able, at present, to settle this question in a satisfactory way. Let us note that the existence of an $n$-dimensional $S$-transform would be equivalent to the existence of an $n$-dimensional "combinatorial Fourier transform on non-crossing partitions", the analogue of the map $\mathcal{F}$ appearing in (b) and (c) of Section 1.3. The $n$-dimensional version of $\mathcal{F}$ should be an isomorphism from, say, $\{f \in \Theta_n \mid [\text{coef } (i)](f) = 1, \ 1 \leq i \leq n\}$, endowed with $\boxed{\star}$, onto some semigroup $\mathcal{M}$, where the operation on $\mathcal{M}$ should be in some way related to the multiplication of formal power series. The various candidates we have tested for $\mathcal{F}$ and $\mathcal{M}$ have all failed to be homomorphic. While, of course, it is not impossible that a more fortunate choice can be found, we would like to remark that some substantial difference between the 1- and multi-dimensional cases is to be expected anyway, in view of the fact that $(\Theta_n, \boxed{\star} \,)$ is *commutative* if and only if $n = 1$. This distinction is particularly significant in the light of the original approach of Voiculescu in [14], where the $S$-transform is found as the exponential of a certain commutative Lie group, the group-operation on which is related to $\boxed{\star}$ on $\Theta_1$. (The analogue of this Lie group in the multidimensional case is non-commutative, and it seems unlikely that a homomorphic $S$-transform could be found by the same method.)

Let us also mention that in a recent work ([3], Part B), U. Haagerup has found another proof of the multiplicativity of the $S$-transform, based on an elegant adaptation of the "Fock space model" for non-commutative random variables. It is possible that the right concept

of multidimensional $S$-transform might be found via a better understanding of this adapted model.

## 4. Proof of the Applications 1.6, 1.10, 1.11, 1.13

**4.1 Notation** Let $n$ be a positive integer. We consider the set $\Theta_n$ of formal power series without constant coefficient in $n$ non-commuting variables $z_1, \ldots, z_n$, and use the notations for coefficients introduced in Section 3.1.

For $f \in \Theta_n$ and a number $r \neq 0$ we will denote by $f \circ D_r$, and call *the dilation of $f$ by $r$,* the series in $\Theta_n$ defined by

$$(f \circ D_r)(z_1, \ldots, z_n) \;=\; f(rz_1, \ldots, rz_n),$$

or equivalently, by

$$[\text{coef } (i_1, \ldots, i_k)](f \circ D_r) \;=\; r^k [\text{coef } (i_1, \ldots, i_k)](f), \tag{4.1}$$

for every $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$.

It is immediate that (4.1) can be extended to

$$[\text{coef } (i_1, \ldots, i_k); \pi](f \circ D_r) \;=\; r^k [\text{coef } (i_1, \ldots, i_k); \pi](f), \tag{4.2}$$

for every $k \geq 1, 1 \leq i_1, \ldots, i_k \leq n$ and $\pi \in NC(k)$. Moreover, from (4.2) and (3.3) it clearly follows that the operation $\boxed{\star}$ discussed in the preceding section behaves nicely under dilations, i.e.

$$(f \circ D_r) \boxed{\star} g \;=\; f \boxed{\star} (g \circ D_r) \;=\; (f \boxed{\star} g) \circ D_r, \tag{4.3}$$

for every $f, g \in \Theta_n$ and $r \neq 0$.

The next lemma will be used in the proofs of 1.6 and 1.10.

**4.2 Lemma** Given a series $f(z) = \sum_{k=1}^{\infty} \alpha_k z^k$ in $\Theta_1$ and a positive integer $n$, let us denote by $f(z_1 + \cdots + z_n)$ the series in $\Theta_n$ determined by

$$[\text{coef } (i_1, \ldots, i_k)](f(z_1 + \cdots + z_n)) \;=\; \text{the coefficient of } z^k \text{ in } f,$$

for every $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$. Then we have that

$$[R(\mu_{\underbrace{a, \ldots, a}_{n}})](z_1, \ldots, z_n) \ = \ [R(\mu_a)](z_1 + \cdots + z_n), \tag{4.4}$$

for every random variable $a$ in a non-commutative probability space $(\mathcal{A}, \varphi)$, and for every $n \geq 1$.

Particular cases: if $b$ is a centered semicircular element of radius $r$ in the noncommutative probability space $(\mathcal{A}, \varphi)$, then:

$$[R(\mu_{\underbrace{b, \ldots, b}_{n}})](z_1, \ldots, z_n) \ = \ \frac{r^2}{4} \sum_{i,j=1}^{n} z_i z_j \in \Theta_n; \tag{4.5}$$

$$R(\mu_{\underbrace{b^2, \ldots, b^2}_{n}}) \ = \ Zeta_n \circ D_{r^2/4} \in \Theta_n. \tag{4.6}$$

**Proof** The analogue of Eqn.(4.4) with the $R$-series replaced by the $M$-series is immediate. It is also immediately checked that

$$(f(z_1 + \cdots + z_n)) \boxed{\star} (g(z_1 + \cdots + z_n)) \ = \ (f \boxed{\star} g)(z_1 + \cdots + z_n) \tag{4.7}$$

for every $f, g \in \Theta_1$, where the $\boxed{\star}$-operations in the left-hand and right-hand side of (4.7) are considered in $\Theta_n$ and $\Theta_1$, respectively. Since from Eqn.(3.11) in 3.8 it is clear that $Moeb_n(z_1, \ldots, z_n) = Moeb_1(z_1 + \cdots + z_n)$, we have:

$$[R(\mu_{a,\ldots,a})](z_1, \ldots, z_n) \ = \ (M(\mu_{a,\ldots,a}) \boxed{\star} Moeb_n)(z_1, \ldots, z_n) \ =$$

$$= \ [M(\mu_a)](z_1 + \cdots + z_n) \boxed{\star} Moeb_1(z_1 + \cdots + z_n)$$

$$\overset{(4.7)}{=} (M(\mu_a) \boxed{\star} Moeb_1)(z_1 + \cdots + z_n) \ = \ [R(\mu_a)](z_1 + \cdots + z_n).$$

Now, if $b$ is centered semicircular of radius $r$ in $(\mathcal{A}, \varphi)$, then $[R(\mu_b)](z) = \frac{r^2}{4} z^2$ (see [17], Example 3.4.4); the formula (4.5) follows from this and (4.4). Similarly, in order to prove (4.6) we only need to show that $[R(\mu_{b^2})](z) = \sum_{k=1}^{\infty} (\frac{r^2}{4} z)^k$. This is equivalent to

$$R(\mu_{b^2}) \boxed{\star} Zeta_1 \ = \ ( \ \sum_{k=1}^{\infty} (\frac{r^2}{4} z)^k \ ) \boxed{\star} Zeta_1. \tag{4.8}$$

By (3.13), the left-hand side of (4.8) is $M(\mu_{b^2})$, i.e. the series

$$\sum_{k=1}^{\infty} \varphi(b^{2k}) z^k \ = \ \sum_{k=1}^{\infty} (\frac{2}{\pi r^2} \int_{-r}^{r} t^{2k} \sqrt{r^2 - t^2} \ dt) z^k.$$

A direct calculation, which takes into account that $|NC(k)| = (2k)!/(k!(k+1)!)$, finds the right-hand side of (4.8) equal to $\sum_{k=1}^{\infty}(r^2/4)^k \cdot (2k)!/(k!(k+1)!) \cdot z^k$; so (4.8) comes to proving that

$$\frac{2}{\pi r^2}\int_{-r}^{r} t^{2k}\sqrt{r^2-t^2}\ dt \ = \ \left(\frac{r^2}{4}\right)^k \cdot \frac{(2k)!}{k!(k+1)!}, \quad k \geq 1,$$

which is easily done via integration by parts and induction on $k$. **QED**

**4.3 Proof of Application 1.6** Let $(\mathcal{A}, \varphi)$ and $a_1, \ldots, a_n, b \in \mathcal{A}$ be as in the statement of 1.6. By using the trace-property of $\varphi$, it is immediately seen that $\mu_{ba_1b,\ldots,ba_nb} = \mu_{a_1b^2,\ldots,a_nb^2}$. Thus:

$$\begin{aligned} R(\mu_{ba_1b,\ldots,ba_nb}) \ &= \ R(\mu_{a_1b^2,\ldots,a_nb^2}) \ \overset{(1.12)}{=} \ R(\mu_{a_1,\ldots,a_n})\boxed{\star}R(\mu_{b^2,\ldots,b^2}) \\ \overset{(4.6)}{=} \ R(\mu_{a_1,\ldots,a_n})\boxed{\star}(Zeta \circ D_{r^2/4}) \ &\overset{(4.3)}{=} \ (R(\mu_{a_1,\ldots,a_n})\boxed{\star}Zeta) \circ D_{r^2/4} \ \overset{(3.13)}{=} \ M(\mu_{a_1,\ldots,a_n}) \circ D_{r^2/4}. \end{aligned}$$

The fact that the latter series can be also written as $M(\mu_{\frac{r^2}{4}a_1,\ldots,\frac{r^2}{4}a_n})$ is immediate. **QED**

We now head towards the proof of 1.11. We will also need to use the behavior of the operation $\boxed{\star}$ under multiplication by a scalar, as described in the following

**4.4 Lemma:** Let $n$ be a positive integer, and let $f, g$ be two series in $\Theta_n$. For every $r \neq 0$ we have

$$(rf)\boxed{\star}(rg) \ = \ (r(f\boxed{\star}g)) \circ D_r. \tag{4.9}$$

**Proof** For every $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$ we have:

$$\begin{aligned} &[\text{coef } (i_1, \ldots, i_k)]((rf)\boxed{\star}(rg)) \\ &\overset{(3.3)}{=} \sum_{\pi \in NC(k)} [\text{coef } (i_1, \ldots, i_k); \pi](rf) \cdot [\text{coef } (i_1, \ldots, i_k); K(\pi)](rg) \\ &\overset{(3.2)}{=} \sum_{\pi \in NC(k)} r^{|\pi|}[\text{coef } (i_1, \ldots, i_k); \pi](f) \cdot r^{|K(\pi)|}[\text{coef } (i_1, \ldots, i_k); K(\pi)](g) \\ &= \ r^{k+1} \sum_{\pi \in NC(k)} [\text{coef } (i_1, \ldots, i_k); \pi](f) \cdot [\text{coef } (i_1, \ldots, i_k); K(\pi)](g) \end{aligned}$$

(because $|\pi| + |K(\pi)| = k+1$ for every $\pi \in NC(k)$)

$$= \ r^{k+1}[\text{coef } (i_1, \ldots, i_k)](f\boxed{\star}g). \tag{4.10}$$

But it is clear that the quantity in (4.10) is exactly the coefficient of $z_{i_1} \cdots z_{i_k}$ in $(r(f \boxed{\star} g)) \circ D_r$. **QED**

**4.5 Remark** It is convenient to use Eqn.(4.9) in the slightly modified form:

$$\frac{1}{r}(f \boxed{\star} (rg)) \;=\; ((\frac{1}{r}f) \circ D_r) \boxed{\star} g, \quad f, g \in \Theta_n, \; r \neq 0; \tag{4.11}$$

(4.11) reduces to (4.9) via the substitution $f = r\widetilde{f}$, $g = \widetilde{g}$.

**4.6 Proof of Application 1.11** Let $(\mathcal{A}, \varphi)$ and $a_1, \ldots, a_n, p \in \mathcal{A}$ be as in the statement of 1.11. Recall that $\alpha \neq 0$ denotes the trace of the idempotent $p$.

It is clear that

$$M(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np}) \;=\; \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{pa_1p,\ldots,pa_np})$$

(since the trace on $p\mathcal{A}p$ is just the restriction of the one on $\mathcal{A}$, normalized by a factor of $1/\alpha$). Also, from the trace property of $\varphi$ it follows immediately that $\mu^{(\mathcal{A})}_{pa_1p,\ldots,pa_np} = \mu^{(\mathcal{A})}_{a_1p,\ldots,a_np}$. Hence, we can write:

$$\begin{aligned} M(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np}) \;&=\; \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{a_1p,\ldots,a_np}) \;\overset{(3.14)}{=}\; \frac{1}{\alpha} \Big( R(\mu^{(\mathcal{A})}_{a_1,\ldots,a_n}) \boxed{\star} M(\mu^{(\mathcal{A})}_{p,\ldots,p}) \Big) \\ &=\; \frac{1}{\alpha} \left( R(\mu_{a_1,\ldots,a_n}) \boxed{\star} (\alpha Zeta) \right) \;\overset{(4.11)}{=}\; ((\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{a_1,\ldots,a_n})) \circ D_\alpha) \boxed{\star} Zeta. \end{aligned}$$

But from $M(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np}) = ((\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{a_1,\ldots,a_n})) \circ D_\alpha) \boxed{\star} Zeta$ we obtain, by $\boxed{\star}$-operating with $Moeb$ on the right:

$$R(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np}) \;=\; M(\mu^{(p\mathcal{A}p)}_{pa_1p,\ldots,pa_np}) \boxed{\star} Moeb \;=\; (\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{a_1,\ldots,a_n})) \circ D_\alpha. \tag{4.12}$$

The fact that the rightmost expression in (4.12) can be also written as $\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{\alpha a_1,\ldots,\alpha a_n})$ is an easy exercise, left to the reader. **QED**

For the proof of the remaining Applications 1.10 and 1.13, we will use the following freeness criterion.

**4.7 Proposition** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space such that $\varphi$ is a trace, let $\mathcal{B} \subseteq \mathcal{A}$ be a unital subalgebra, and let $\mathcal{X} \subseteq \mathcal{A}$ be a non-void subset. Assume that for every $m \geq 1$ and $b_1, \ldots, b_m \in \mathcal{B}$, $x_1, \ldots, x_m \in \mathcal{X}$, it is true that

$$\varphi(b_1 x_1 b_2 x_2 \cdots b_m x_m) \;=\; [\mathrm{coef}\ (1, 2, \ldots, m)](R(\mu_{b_1,\ldots,b_m}) \boxed{\star} M(\mu_{x_1,\ldots,x_m})) \tag{4.13}$$

(where, according to the notations set in 3.1, the right-hand side of (4.13) denotes the coefficient of $z_1 z_2 \cdots z_m$ in the formal power series $R(\mu_{b_1,\ldots,b_m}) \boxed{\star} M(\mu_{x_1,\ldots,x_m}) \in \Theta_m$). Then $\mathcal{X}$ is free from $\mathcal{B}$ in $(\mathcal{A}, \varphi)$.

Note that the condition expressed in (4.13) is also necessary for the freeness of $\mathcal{X}$ from $\mathcal{B}$, as it clearly follows from Theorem 1.4 and Remark 3.10 (and the obvious fact that $\varphi(b_1x_1b_2x_2\cdots b_mx_m) = [\text{coef } (1,2,\ldots,m)](M(\mu_{b_1x_1,\ldots,b_mx_m}))$ ).

**Proof** Let $\mathcal{C}$ be the unital subalgebra of $\mathcal{A}$ generated by $\mathcal{X}$. The fact that $\mathcal{X}$ is free from $\mathcal{B}$ means, by definition, that the subalgebras $\mathcal{B}$ and $\mathcal{C}$ are free in $(\mathcal{A}, \varphi)$. We consider the free product of unital algebras $\mathcal{B} \star \mathcal{C}$ (its construction is independent of the fact that $\mathcal{B}$ and $\mathcal{C}$ lie in the same algebra $\mathcal{A}$); we will denote the canonical embeddings of $\mathcal{B}$ and $\mathcal{C}$ in $\mathcal{B} \star \mathcal{C}$ by $b \to \widetilde{b}$ and $c \to \widetilde{c}$, respectively (that is, $\widetilde{b}$ "is the name" of the element $b \in \mathcal{B}$, when viewed in $\mathcal{B} \star \mathcal{C}$, and similarly for $c \in \mathcal{C}$ and $\widetilde{c} \in \mathcal{B} \star \mathcal{C}$). By the universality of $\mathcal{B} \star \mathcal{C}$, there exists a unique homomorphism of unital algebras $\Phi : \mathcal{B} \star \mathcal{C} \to \mathcal{A}$, such that $\Phi(\widetilde{b}) = b$ for every $b \in \mathcal{B}$ and $\Phi(\widetilde{c}) = c$ for every $c \in \mathcal{C}$ (see e.g. [17], Section 1.2).

Now, let us consider on $\mathcal{B} \star \mathcal{C}$ the free product functional $\psi = (\varphi|\mathcal{B}) \star (\varphi|\mathcal{C})$; this is a trace, because $\varphi|\mathcal{B}$ and $\varphi|\mathcal{C}$ are so and by Proposition 2.5.3 of [17]. We remark that:

(a) We have

$$\varphi(b_1x_1\cdots b_mx_m) \;=\; \psi(\widetilde{b}_1\widetilde{x}_1\cdots \widetilde{b}_m\widetilde{x}_m), \tag{4.14}$$

for every $m \geq 1,\ b_1, \ldots b_m \in \mathcal{B},\ x_1, \ldots, x_m \in \mathcal{X}$. (4.14) holds because both its sides are equal to:

$$[\text{coef } (1,2,\ldots,m)](R(\mu_{b_1,\ldots,b_m}) \boxed{\star} M(\mu_{x_1,\ldots,x_m})). \tag{4.15}$$

The equality between (4.15) and the left-hand side of (4.14) is ensured by the hypothesis, while the equality between (4.15) and the right-hand side of (4.14) comes out from Theorem 1.4 in the form presented in Remark 3.10 (by also taking into account that $\{\widetilde{b} \mid b \in \mathcal{B}\}$ and $\{\widetilde{c} \mid c \in \mathcal{C}\}$ are free in $(\mathcal{B} \star \mathcal{C}, \psi)$, and the obvious fact that $\mu_{\widetilde{b}_1,\ldots,\widetilde{b}_m} = \mu_{b_1,\ldots,b_m}$, $\mu_{\widetilde{x}_1,\ldots,\widetilde{x}_m} = \mu_{x_1,\ldots,x_m}$).

(b) (4.14) implies that $\varphi \circ \Phi = \psi$. Indeed, since $\Phi(\widetilde{b}_1\widetilde{x}_1\cdots \widetilde{b}_m\widetilde{x}_m) = b_1x_1\cdots b_mx_m$, (4.14) is verifying the coincidence of the functionals $\varphi \circ \Phi$ and $\psi$ on elements of the form $\widetilde{b}_1\widetilde{x}_1\cdots \widetilde{b}_m\widetilde{x}_m$. Since $\varphi \circ \Phi$ and $\psi$ are traces, their coincidence also follows on elements of the form $\widetilde{b}_1\widetilde{x}_1\cdots \widetilde{b}_m\widetilde{x}_m\widetilde{b}_{m+1}$, with $m \geq 0$ (if $m = 0$ this is clear, if $m \geq 1$ we send $\widetilde{b}_{m+1}$ to the front and multiply it with $\widetilde{b}_1$). But the elements of these two forms are spanning linearly together all of $\mathcal{B} \star \mathcal{C}$ (first, it is clear that $\mathcal{B} \star \mathcal{C}$ is spanned linearly by products of $\widetilde{b}$'s and

$\widetilde{x}$'s; then any such product can be turned into an alternating one, by multiplying together the consecutive $b$'s and by inserting an $\widetilde{1}$, $1 \in \mathcal{B}$, between consecutive $\widetilde{x}$'s; also, by adding if necessary an $\widetilde{1}$ on the left, it may be always assumed that the monomial starts with a $\widetilde{b}$).

(c) The freeness of $\mathcal{B}$ and $\mathcal{C}$ in $(\mathcal{A}, \varphi)$ is an immediate consequence of the fact that $\varphi \circ \Phi = \psi$. Indeed, let us consider an alternating sequence of elements from $\mathcal{B}$ and $\mathcal{C}$, e.g. $c_1, b_1, \ldots, c_n, b_n, c_{n+1}$, such that $\varphi(c_1) = \varphi(b_1) = \cdots = \varphi(c_n) = \varphi(b_n) = \varphi(c_{n+1}) = 0$. Then, clearly, we also have $\psi(\widetilde{c}_1) = \varphi(c_1) = 0$, $\psi(\widetilde{b}_1) = \varphi(b_1) = 0$, $\ldots$, $\psi(\widetilde{c}_{n+1}) = \varphi(c_{n+1}) = 0$. From the freeness of $\{\widetilde{b} \mid b \in \mathcal{B}\}$ and $\{\widetilde{c} \mid c \in \mathcal{C}\}$ in $(\mathcal{B} \star \mathcal{C}, \psi)$ it follows that $\psi(\widetilde{c}_1 \widetilde{b}_1 \cdots \widetilde{c}_n \widetilde{b}_n \widetilde{c}_{n+1}) = 0$; but "$\varphi \circ \Phi = \psi$" implies that $\psi(\widetilde{c}_1 \widetilde{b}_1 \cdots \widetilde{c}_n \widetilde{b}_n \widetilde{c}_{n+1}) = \varphi(c_1 b_1 \cdots c_n b_n c_{n+1})$, so the latter quantity is also vanishing. **QED**

**4.8 Remark** The preceding proposition could be equivalently stated by replacing (4.13) with

$$\varphi(b_1 x_1 b_2 x_2 \cdots b_m x_m) \;=\; [\text{coef } (1, 2, \ldots, m)](M(\mu_{b_1, \ldots, b_m}) \boxed{\star} R(\mu_{x_1, \ldots, x_m})). \tag{4.16}$$

The proof would be exactly the same, with the only detail that at the point where Remark 3.10 is invoked (in part (a) of the proof), we would now refer to Eqn.(3.15) instead of (3.14).

**4.9 The proof of Application 1.13** Let $(\mathcal{A}, \varphi)$, $a_1, \ldots, a_n \in \mathcal{A}$, $p \in \mathcal{B} \subseteq \mathcal{A}$ be as in the statement of 1.13. We will use the criterion established in 4.7 for proving the freeness of $\mathcal{X} = \{pa_1 p, \ldots, pa_n p\}$ from the subalgebra $p\mathcal{B}p$ in the non-commutative probability space $(p\mathcal{A}p, \frac{1}{\alpha}\varphi | p\mathcal{A}p)$. We fix $m \geq 1$ and $pb_1 p, \ldots, pb_m p \in p\mathcal{B}p$ (with $b_1, \ldots, b_m \in \mathcal{B}$), $x_1, \ldots, x_m \in \mathcal{X}$, about which we want to verify the equality (4.13) of 4.7. Writing explicitly $x_1 = pa_{i_1} p, \ldots, x_m = pa_{i_m} p$ (for some $1 \leq i_1, \ldots i_m \leq n$), the desired equality comes to:

$$\frac{1}{\alpha}\varphi((pb_1 p)(pa_{i_1} p) \cdots (pb_m p)(pa_{i_m} p)) \;=\; \tag{4.17}$$

$$[\text{coef } (1, 2, \ldots, m)] \left( R(\mu^{(p\mathcal{A}p)}_{pb_1 p, \ldots, pb_m p}) \boxed{\star} M(\mu^{(p\mathcal{A}p)}_{pa_{i_1} p, \ldots, pa_{i_m} p}) \right).$$

The two power series involved in the right-hand side of (4.17) can be re-written as follows:

$$R(\mu^{(p\mathcal{A}p)}_{pb_1 p, \ldots, pb_m p}) \;=\; M(\mu^{(p\mathcal{A}p)}_{pb_1 p, \ldots, pb_m p}) \boxed{\star} Moeb \;=\; \left( \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{pb_1 p, \ldots, pb_m p}) \right) \boxed{\star} Moeb, \tag{4.18}$$

and

$$M(\mu^{(p\mathcal{A}p)}_{pa_{i_1} p, \ldots, pa_{i_m} p}) \;=\; \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{pa_{i_1} p, \ldots, pa_{i_m} p}) \;=\; \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{a_{i_1} p, \ldots, a_{i_m} p})$$

$$= \frac{1}{\alpha}\left( R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}}) \boxed{\star} M(\mu^{(\mathcal{A})}_{p,\ldots,p}) \right) \quad \text{(by Theorem 1.4 and Remark 3.10)}$$

$$= \frac{1}{\alpha}\left( R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}}) \boxed{\star} (\alpha Zeta) \right) \overset{(4.11)}{=} \left( (\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}})) \circ D_\alpha \right) \boxed{\star} Zeta. \tag{4.19}$$

When $\boxed{\star}$-multiplying together the rightmost expressions in (4.18) and (4.19), the factors $Moeb$ and $Zeta$ cancel (because $Moeb$ and $Zeta$ are central, and inverse to each other), and we obtain:

$$\begin{aligned} & \left( \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{pb_1p,\ldots,pb_mp}) \right) \boxed{\star} \left( (\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}})) \circ D_\alpha \right) \\ \overset{(4.3)}{=} & \left( (\frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{pb_1p,\ldots,pb_mp})) \boxed{\star} (\frac{1}{\alpha} R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}})) \right) \circ D_\alpha \\ \overset{(4.9)}{=} & \left( \frac{1}{\alpha} (\ M(\mu^{(\mathcal{A})}_{pb_1p,\ldots,pb_mp}) \boxed{\star} R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}})\ ) \circ D_{1/\alpha} \right) \circ D_\alpha \\ = & \frac{1}{\alpha} \left( M(\mu^{(\mathcal{A})}_{pb_1p,\ldots,pb_mp}) \boxed{\star} R(\mu^{(\mathcal{A})}_{a_{i_1},\ldots,a_{i_m}}) \right) \\ = & \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{(pb_1p)a_{i_1},\ldots,(pb_mp)a_{i_m}}), \end{aligned}$$

where at the last equality sign in the sequence we used again the Theorem 1.4 in the form presented in Remark 3.10 (and applied to the instance "$\{a_1, \ldots, a_n\}$ free from $\mathcal{B}$"). Hence the right-hand side of (4.17) has become:

$$[\text{coef } (1, 2, \ldots, m)] \left( \frac{1}{\alpha} M(\mu^{(\mathcal{A})}_{(pb_1p)a_{i_1},\ldots,(pb_mp)a_{i_m}}) \right),$$

and it is immediately verified that the left-hand side of (4.17) is exactly the same thing. **QED**

Finally, for the proof of Application 1.10 we will need the following

**4.10 Lemma:** Let $m$ be a positive integer, and let $c_1, \ldots, c_m, c'_1, \ldots, c'_m$ be random variables in some non-commutative probability space $(\mathcal{A}, \varphi)$. Then

$$[\text{coef } (1, \ldots, m)](M(\mu_{c_1,\ldots,c_m}) \boxed{\star} (M(\mu_{c'_1,\ldots,c'_m})) = \tag{4.20}$$

$$= [\text{coef } (1, 2, \ldots, 2m)] \left( M(\mu_{c_1,c'_1,\ldots c_m,c'_m}) \boxed{\star} Sqsum \right),$$

where $Sqsum \in \Theta_{2m}$ is the quadratic polynomial

$$Sqsum(z_1, z_2, \ldots, z_{2m}) = \sum_{i,j=1}^{2m} z_i z_j. \tag{4.21}$$

**Proof** We have, by the definition,

$$[\text{coef } (1,2,\ldots,2m)] \left( M(\mu_{c_1,c_1',\ldots c_m,c_m'}) \boxed{\star} Sqsum \right) =$$

$$\sum_{\sigma \in NC(2m)} [\text{coef } (1,2,\ldots,2m);\sigma](M(\mu_{c_1,c_1',\ldots,c_m,c_m'})) \cdot [\text{coef } (1,2,\ldots,2m); K(\sigma)](Sqsum). \tag{4.22}$$

A partition $\sigma \in NC(2m)$ can bring a non-zero contribution to the sum (4.22) only if $K(\sigma)$ is a pairing (otherwise the coefficient of $Sqsum$ vanishes), so (4.22) is in fact equal to

$$\sum_{\substack{\sigma \in NC(2m) \\ \textit{such that} \\ K(\sigma) \in NCP(2m)}} [\text{coef } (1,2,\ldots,2m);\sigma](M(\mu_{c_1,c_1',\ldots,c_m,c_m'})). \tag{4.23}$$

But, in view of the Proposition in 2.6, a partition $\sigma \in NC(2m)$ has $K(\sigma) \in NCP(2m)$ if and only if there exists a $\rho \in NC(m)$ (uniquely determined) such that $\sigma$ is obtained by letting $\rho$ work on $\{1,3,\ldots,2m-1\}$ and by letting $K(\rho)$ work on $\{2,4,\ldots,2m\}$. Moreover, if $\sigma \in NC(2m)$ and $\rho \in NC(m)$ are related in this way, it is immediate that

$$[\text{coef } (1,2,\ldots,2m);\sigma](M(\mu_{c_1,c_1',\ldots,c_m,c_m'})) =$$

$$[\text{coef } (1,\ldots,m);\rho](M(\mu_{c_1,\ldots,c_m})) \cdot [\text{coef } (1,\ldots,m);K(\rho)](M(\mu_{c_1',\ldots,c_m'})).$$

This observation shows that the sum (4.23) can be re-written as

$$\sum_{\rho \in NC(m)} [\text{coef } (1,\ldots,m);\rho](M(\mu_{c_1,\ldots,c_m})) \cdot [\text{coef } (1,\ldots,m);K(\rho)](M(\mu_{c_1',\ldots,c_m'}));$$

the latter quantity is just the expansion of the left-hand side of (4.20), so we are done. **QED**

**4.11 The proof of Application 1.10** Let $(\mathcal{A},\varphi)$ and $a_1,\ldots,a_n,b \in \mathcal{A}$ be as in the statement of (1.10). Let $\mathcal{C}$ denote the unital subalgebra of $\mathcal{A}$ generated by $a_1,\ldots,a_n$, and put $\mathcal{X} = \{ba_1b,\ldots,ba_nb\}$. We will use the freeness criterion 4.7, in the form mentioned in Remark 4.8, for proving that $\mathcal{X}$ is free from $\mathcal{C}$ in $(\mathcal{A},\varphi)$. The sufficient condition (4.16) of 4.8 becomes here:

$$\varphi(c_1(ba_{i_1}b)c_2(ba_{i_2}b)\cdots c_m(ba_{i_m}b)) = [\text{coef } (1,2,\ldots,m)] \left( M(\mu_{c_1,\ldots,c_m}) \boxed{\star} R(\mu_{ba_{i_1}b,\ldots,ba_{i_m}b}) \right), \tag{4.24}$$

to be verified for every $m \geq 1$, $c_1,\ldots,c_m \in \mathcal{C}$ and $1 \leq i_1,\ldots,i_m \leq n$.

The left-hand side of (4.24) is equal to [coef $(1,2,\ldots,2m)](M(\mu_{c_1 b, a_{i_1} b, \ldots, c_m b, a_{i_m} b}))$. But since $b$ is free from $\mathcal{C}$ (which contains $c_1,\ldots,c_m,a_{i_1},\ldots,a_{i_m}$), we have, by Theorem 1.4 and Remark 3.10:

$$M(\mu_{c_1 b, a_{i_1} b, \ldots, c_m b, a_{i_m} b}) \;=\; M(\mu_{c_1, a_{i_1}, \ldots, c_m, a_{i_m}}) \boxed{\star} R(\mu_{\underbrace{b,\ldots b}_{2m}}).$$

As it was pointed out in Lemma 4.2, $R(\mu_{\underbrace{b,\ldots b}_{2m}})$ is exactly $Sqsum \circ D_{r/2}$, where $r$ is the radius of $b$ and $Sqsum \in \Theta_{2m}$ is the series appearing in Eqn.(4.21) of Lemma 4.10. Hence (by also using (4.3)):

$$M(\mu_{c_1 b, a_{i_1} b, \ldots, c_m b, a_{i_m} b}) \;=\; \Big(M(\mu_{c_1, a_{i_1}, \ldots, c_m, a_{i_m}}) \boxed{\star} Sqsum\Big) \circ D_{r/2},$$

and the left-hand side of (4.24) becomes

$$\left(\frac{r}{2}\right)^{2m} \cdot [\text{coef } (1,2,\ldots,2m)] \Big(M(\mu_{c_1, a_{i_1}, \ldots, c_m, a_{i_m}}) \boxed{\star} Sqsum\Big) \;=$$

$$\overset{Lemma\ 4.10}{=} \left(\frac{r}{2}\right)^{2m} \cdot [\text{coef } (1,\ldots,m)] \Big(M(\mu_{c_1,\ldots,c_m}) \boxed{\star} M(\mu_{a_{i_1},\ldots,a_{i_m}})\Big). \tag{4.25}$$

On the other hand, we know from Application 1.6 that $R(\mu_{b a_{i_1} b, \ldots, b a_{i_m} b}) = M(\mu_{a_{i_1},\ldots,a_{i_m}}) \circ D_{r^2/4}$; consequently, we have

$$M(\mu_{c_1,\ldots,c_m}) \boxed{\star} R(\mu_{b a_{i_1} b, \ldots, b a_{i_m} b}) \;=\; \Big(M(\mu_{c_1,\ldots,c_m}) \boxed{\star} M(\mu_{a_{i_1},\ldots,a_{i_m}})\Big) \circ D_{r^2/4},$$

and this makes clear that the right-hand side of (4.24) is also equal to (4.25). **QED**

## References


[1] H. Bercovici, D. Voiculescu. Superconvergence to the central limit and failure of the Cramer Theorem for free random variables, preprint.

[2] Ph. Biane. Some properties of crossings and partitions, preprint.

[3] U. Haagerup. On addition and multiplication of free random variables, manuscript.

[4] G. Kreweras. Sur les partitions non-croisées d'un cycle, Discrete Math. 1(1972), 333-350.

[5] A. Nica. $R$-transforms of free joint distributions, and non-crossing partitions, to appear in the Journal of Functional Analysis.

[6] A. Nica, R. Speicher. A "Fourier transform" for multiplicative functions on non-crossing partitions, preprint.

[7] R. Simion, D. Ullman. On the structure of the lattice of non-crossing partitions, Discrete Math. 98(1991), 193-206.

[8] R. Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution, Math. Annalen 298(1994), 611-628.

[9] R. Speicher. Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, Habilitationsschrift, Heidelberg 1994.

[10] R.P. Stanley. Enumerative combinatorics, volume I, Wadsworth & Brooks / Cole Mathematics Series, 1986.

[11] J. Touchard. Sur un problème de configurations et sur les fractions continues, Canadian J. Math. 4(1952), 2-25.

[12] D. Voiculescu. Symmetries of some reduced free product C*-algebras, in Operator algebras and their connection with topology and ergodic theory, H. Araki et al. editors (Springer Lecture Notes in Mathematics, volume 1132, 1985), 556-588.

[13] D. Voiculescu. Addition of certain non-commuting random variables, J. Funct. Anal. 66(1986), 323-346.

[14] D. Voiculescu. Multiplication of certain non-commuting random variables, J. Operator Theory 18(1987), 223-235.

[15] D. Voiculescu. Circular and semicircular systems and free product factors, in Operator algebras, unitary representations, enveloping algebras, and invariant theory, A. Connes et al. editors, Birkhäuser, 1990.

[16] D. Voiculescu. Free probability theory: random matrices and von Neumann algebras, preprint.

[17] D. Voiculescu, K. Dykema, A. Nica. Free random variables, CRM Monograph Series, volume 1, AMS, 1992.